Fuzzy Dynamical Systems in SIR and HIV Models

A Thesis

Submitted by

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Under the supervision

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DECLARATION

I declare that the topic **"Fuzzy Dynamical Systems in SIR and HIV Models"** for completion for my master degree has not been submitted in any other institution or University for the award of any other degree or diploma.

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CERTIFICATE

This is to certify that the project report entitled **"Fuzzy Dynamical Systems in SIR and HIV Models"** submitted by **Abhay Kumar** to the National Institute of Technology Rourkela, Orissa for the partial fulfilment of requirements for the degree of Master of Science in Mathematics is a bonafide record of review work carried out by him under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

May, 2015

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ABSTRACT

In this report I applied Homotopy analysis method to solve SIR Model and HIV model using fuzzy initial boundary value problem. In order to solve ordinary differential equation we use a Zeroth order deformation equation which relates between linear differential equation to nonlinear differential equation. And then apply the Taylor series concept to change the equation into iterative form. We convert the HIV model into two crisp differential fuzzy model and after that we used Homotopy analysis method to solve HIV model using given initial condition. Homotopy analysis method is easy-to-use analytic method to solve fuzzy initial value problems.

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Introduction

In my thesis, series solution of fuzzy initial value problems of differentiability by means of the Homotopy analysis method is considered. The new approach provides the solution in the form of a rapidly convergent series with easily computable components using symbolic computation software. The Homotopy analysis method contains the auxiliary parameter, the convergence region of the series solution can be controlled in a simple way. The proposed technique is applied to a few test examples to illustrate the accuracy and applicability of the method. The results reveal that the method is very effective and straightforward. Meanwhile, obtain results show that the Homotopy analysis method is a powerful and easy-to-use and to solve fuzzy initial value problems.

CHAPTER 1

Theory behind solving fuzzy initial value problem

In this section we define a first order fuzzy initial value problem (FIVP) of differentiability than we replace it by its parametric form. And after that we present an algorithm to solve the new system which consist of two ODEs.

Let us consider ordinary differential equation

$$x'(t) = f\big(t, x(t)\big), \qquad t_0 \leq t \leq t_0 + a, \qquad where\ a > 0 \tag{1.1}$$

and initial condition $x(t_0) = x^0$,

(1.2)

where $f: [t_0, t_0 + a\ ] \times R \to R$ is a continuous real –valued function, $x^0 \in R$ and t_0 and a are finite constant with $a > 0$.

Now in order to solve above equation simultaneously we write the fuzzy function $x(t)$in its *r*-cut representation to get

$[x(t)]^r = \big[\underline{x}_r(t), \overline{x}_r(t)\big]$,

and initial condition in *r* cut form

$[x(0)]^r = \big[\underline{x}_r(0), \overline{x}_r(0)\big]\ or\ \big[\underline{x}_r^0\ , \overline{x}_r^0\big]$.

The extension principal of Zadeh leads to the following definition of $f\big(t, x(t)\big)$ where $x(t)$ is a fuzzy number

$f\big(t, x(t)\big)(s) = \sup\{x(t)(\mathrm{r})\ = f(t, \mathrm{r}), s \in R\}$.

According to Nguyen theorem it follows that

$$\begin{aligned}[f(t, x(t))]^r &= \Big[\underline{f}_r\big(t, x(t)\big), \overline{f}_r\big(t, x(t)\big)\Big] = f(t, [x(t)]^r) \\ &= \{f(t, y): y \in [x(t)]^r\} \\ &= \Big[f_{1,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right), f_{2,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right)\Big],\end{aligned}$$

where the two term end point function are given as

$$f_{1,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right) = \min\{f(t, y): y \in [x(t)]^r\},$$

$$f_{2,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right) = \max\{f(t, y): y \in [x(t)]^r\}.$$

Definition: let $[t_0, t_0 + a\ \ ]$ and $R \to R_f$ such that $D_1 x$ or $D_2 x$ exist. If x and $D_1 x$ satisfy FIVP (1.1) and (1.2), we say x is a (1.1) solution of FIVP (1.1) and (1.2). Similarly, if x and $D_2 x$ satisfy FIVP (1.1) and (1.2), we say that x is a (1.2) - solution of FIVP (1.1) and (1.2).

The objective of the next algorithm is to follow a method to solve (1.1) and (1.2) in parametric form in terms of its r-cut representation.

Algorithm

Case 1: if x (t) is (1)-differentiable then $[D_1 x(t)]^r = \left[\underline{x}_r'(t), \overline{x}_r'(t)\right]$ and solving FIVP (1) and (2) translate into the following subroutine

Step (i): solve the following system of ODEs for $\left[\underline{x}_r(t), \overline{x}_r(t)\ \right]$

$$\begin{cases} \underline{x}_r'(t) = f_{1,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right), \\ \underline{x}_r'(t) = f_{2,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right), \end{cases} \tag{1.3}$$

subject to the initial condition

$$\begin{cases} \underline{x}_r(t_0) = \underline{x}_r^0, \\ \overline{x}_r(t_0) = \overline{x}_r^0, \end{cases} \tag{1.4}$$

Step (ii): ensure that the solution $\underline{x}_r(t), \overline{x}_r(t)$ and its derivative $\left[\underline{x}_r'(t), \overline{x}_r'(t)\right]$are valid level sets for each other r∈ [0, 1],

Step (iii): use equation $u(s) = \sup\{r\colon \underline{u}(r) \le s \le \overline{u}(r)\}$ to construct a (1) solution x(t) such that $[x(t)]^r = \underline{x}_r(t), \overline{x}_r(t)$ for each r∈[0,1].

Case 2: If x (t) is two differentiable the $[D_2 x(t)]^r = \left[\overline{x}_r'(t), \underline{x}_r'(t)\right]$ and solving FIVP (1.1) and (1.2) and translate into the following routine:

Step (i): solve the following system of ODEs for $\underline{x}_r(t), \overline{x}_r(t)$

$$\begin{cases} \underline{x}_r'(t) = f_{2,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right)\ , \\ \underline{x}_r'(t) = f_{1,r}\left(t, \underline{x}_r(t), \overline{x}_r(t)\right)\ , \end{cases} \tag{1.5}$$

Subject to the initial condition

$$\begin{cases} \underline{x}_r(t_0) = \underline{x}_r^0\ , \\ \overline{x}_r(t_0) = \overline{x}_r^0\ , \end{cases} \tag{1.6}$$

Step (ii): ensure that the solution $\underline{x}_r(t), \overline{x}_r(t)$ and its derivative $\left[\underline{x}_r'(t), \overline{x}_r'(t)\right]$are valid level sets for each other r∈ [0, 1],

Step (iii): use equation $u(s) = \sup\{r: \underline{u}(r) \leq s \leq \overline{u}(r)\}$ to construct a (1.1) solution $x(t)$ such that $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each r∈ [0, 1].

Sometimes we can not decompose the membership function of the solution $[x(t)]^r$ as a function define on R for each $t \in [t_0, t_0 + a]$ Then from (1.3) we stop the solution in term of r-cut representation.

Basic idea of Homotopy analysis method (HAM)

Homotopy analysis method is used to solve differential equation. To achieve our goal we consider the nonlinear differential equation

$$N[x(t)] = 0, t \geq t_0, \tag{1.7}$$

where N is a nonlinear differential operator and $x(t)$ is an unknown function of the independent variable t.

Liao create the Zeroth order deformation equation

$$(1-q)\mathcal{L}[\Phi(t;q) - x_0(t)] = q\hbar H(t)N[\Phi(t;q)]\,, \tag{1.8}$$

where

$q \in [0, 1]$ →embedding parameter,

$\hbar \neq 0$ → auxiliary parameter,

$H(t) \neq 0$ → auxiliary function,

N → nonlinear differential operator,

$\Phi(t;q)$ → unknown function,

$x_0(t)$ → Initial guess of x (t) which satisfy the initial condition,

$\mathcal{L}$ → auxiliary linear operator with the property,

$$\mathcal{L}\,[f(t)] = 0 \text{ when } f(t) = 0. \tag{1.9}$$

It should be emphasized that one has a great freedom to choose the initial guess $x_0(t)$, the auxiliary linear operator $\mathcal{L}$, auxiliary parameter ħ and the auxiliary function $H(t)$. According to the property (1.7) and the suitable initial condition when q=0 we have

$$\Phi(t;q) = x_0(t), \tag{1.10}$$

and when q=1, since $\hbar \neq 0$ and $H(t) \neq 0$, the zeroth-order deformation equation (1.8) is equivalent to equation (1.7), hence

$$\Phi(t;1) = x(t). \tag{1.11}$$

Thus according to equation (1.10) and (1.11), as q increasing from 0 to 1, the solution $\Phi(t;q)$ various continuously from the initial approximation $x_0(t)$ to the exact solution $x(t)$.

Define the m^{th}- order deformation derivative

$$x_m(t) = \frac{1}{m!}\frac{\partial^m \Phi(t;q)}{\partial q^m} \;, \tag{1.12}$$

expanding $\Phi(t;q)$ in a Taylor series with respect to the embedding parameter q,

by using equation (1.10) and (1.12) we have

$$\Phi(t;q) = x_0(t) + \sum_{m=1}^{\infty} x_m(t) q^m \;. \tag{1.13}$$

Assume that the auxiliary parameter *ħ*, the auxiliary function *H* (t), the initial approximation $x_0(t)$, and the auxiliary linear operator *£* are properly chosen so that the series (1.13) of $\Phi(t;q)$ converges at *q=1.* Then we have these assumption the series solution

$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t).$

Define the vector

$\vec{x}_n(t) = \{x_0(t), x_1(t), \dots, x_n(t)\}.$

Differentiating equation (1.8) *m*-times with respect to embedding parameter *q*, and then setting *q=0* and finally dividing them by *m!* We have, using equation

eq. (1.12), the so m^{th}-order deformation equation

$$\mathcal{L}[x_m(t) - \chi_m x_{m-1}(t)] = \hbar H(t) \boldsymbol{\Re_m}(\vec{x}_{m-1}(t)) \;, \tag{1.14}$$

where $\mathrm{m} = 1,2, \dots, \mathrm{n}, \chi_m = \begin{cases} 0\,, m \le 1, \\ 1\,,\; m > 1, \end{cases}$ and

$$\boldsymbol{\Re_k}\big(\vec{x}_{k-1}(t)\big) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\Phi(t;q)]}{\partial p^{m-1}}\bigg|_{q=0}. \tag{1.15}$$

For given any nonlinear operator *N*, the term $\boldsymbol{\Re_m}(\vec{x}_{m-1}(t))$ can be easily expressed by eqs. (1.15) and (1.18). Thus we can gain $x_0(t), x_1(t), \dots, x_n(t)$ by means of solving the linear high-order deformation eqs. (1.14) and (17) one after the other in order. The m^{th} order approximation of *x (t)* is given by $x(t) = \sum_{k=1}^{m-1} x_k(t)$.

It should be emphasized that the so-called m^{th} order deformation equations (1.14) is linear.

CHAPTER 2-SIR MODEL

Model -The problem of spreading of a non-fatal disease in a population which is assume to have a constant size over the period of the epidemic is considered.

At time t suppose the population consist of

$$\begin{cases} x(t) \to uninfected\ liable\ to\ infected\ , \\ y(t) \to infected\ means\ who\ have\ the\ diseases\ , \\ z(t) \to isolated\ who\ have\ immune\ and. \end{cases}$$

Assume there is a steady rate constant between susceptible and infected and that a constant proportion of these constant result in transmission. Then in time $\delta t, x$ of the susceptible become infected.

$$\delta x = -\beta xy\delta t \tag{2.1}$$

And β is a positive constant. If $\gamma > 0$ is a rate at which current infective become isolated, then

$$\delta y = \beta xy\delta t - \gamma y\delta t. \tag{2.2}$$

The number of new isolated δz is given by

$$\delta z = \gamma y\delta t. \tag{2.3}$$

Now let $\delta t \to \infty$. Then the following system determine the progress of the disease.

The classic SIR epidemic model is given by the following system of nonlinear ordinary differential equation:

$$\begin{cases} \frac{dx}{dt} = -\beta xy \\ \frac{dy}{dt} = \beta xy - \gamma y \\ \frac{dz}{dt} = \gamma y \end{cases} \tag{2.4}$$

with initial condition,

$x(0) = N_1,\ y(0) = N_2,\ z(0) = N_3.$

and m^{th}-order deformation equation

$$\mathfrak{R}_{\boldsymbol{k}}\left(\vec{x}_{k-1}(t)\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1}N[\Phi(t;p)]}{\partial q^{k-1}}\bigg|_{p=0}. \tag{2.5}$$

Now applying the inverse of the operator

$\frac{d(.)}{dt} \to \int_0^t (.)dt$ to each equation in the system (2.4) we will get,

$$\begin{cases} x(t) = x(0) - \int_0^t \beta x(t)y(t)dt\ , \\ y(t) = y(0) + \int_0^t [\beta x(t)y(t) - \gamma y(t)]dt\ , \\ z(t) = z(0) + \int_0^t \gamma y(t)dt\ . \end{cases} \tag{2.6}$$

After applying zeroth order on equation (2.5)

$$\begin{cases}(1-p)\big(X(t,p)-x_0(t)\big)=\hbar p\left(X(t,p)-x(0)+\int_0^t \beta x(t)y(t)dt\right) \quad , \\ (1-p)\big(Y(t,p)-y_0(t)\big)=\hbar p\left(Y(t,p)-y(0)+\int_0^t [\beta x(t)y(t)-\gamma y(t)]dt\right), \\ (1-p)\big(Z(t,p)-z_0(t)\big)=\hbar p\left(Z(t,p)-z(0)-\int_0^t \gamma y(t)dt\right),\end{cases} \tag{2.7}$$

and replacing $p=0$ and $p=1$ we will get respectively

$$\begin{cases}X(t,0)=x_0(t)\\ Y(t,0)=y_0(t)\\ Z(t,0)=z_0(t)\end{cases} \qquad \begin{cases}X(t,1)=x(t),\\ Y(t,1)=y(t),\\ Z(t,1)=z(t).\end{cases}$$

According to given Taylor series with respect to q

$\Phi(t,q)=x_0(t)+\sum_{m=1}^{\infty} x_m(t)q^m,$

Based on above series the value of X, Y, Z will become ,

$$\begin{cases}X(t,p)=x_0(t)+\sum_{k=1}^{\infty} x_k(t)p^k ,\\ Y(t,p)=x_0(t)+\sum_{k=1}^{\infty} y_m(t)p^k ,\\ Z(t,p)=z_0(t)+\sum_{k=1}^{\infty} z_m(t)p^k .\end{cases} \tag{2.8}$$

If $p=1$ the using equation (2.6) we will get

$$\begin{cases}X(t,1)=x(t)+\sum_{k=1}^{\infty} x_k(t),\\ y(t,1)=y_0(t)+\sum_{k=1}^{\infty} y_k(t),\\ z(t,1)=z_0(t)+\sum_{k=1}^{\infty} z_k(t).\end{cases} \tag{2.9}$$

Define the vectors

$$\begin{cases}\vec{x}_k=\{x_0(t),x_1(t),\dots,x_k(t)\}\\ \vec{y}_k=\{y_0(t),y_1(t),\dots,y_k(t)\}\\ \vec{z}_k=\{z_0(t),z_1(t),\dots,z_k(t)\}\end{cases} \tag{2.10}$$

so k^{th} deformation equation will become

$$\begin{cases}\mathcal{L}[x_k-\chi_k x_{k-1}(t)]=\hbar\Re_k(\vec{x}_{k-1}),\\ \mathcal{L}[y_k-\chi_k y_{k-1}(t)]=\hbar\Re_k(\vec{y}_{k-1}),\\ \mathcal{L}[z_k-\chi_k z_{k-1}(t)]=\hbar\Re_k(\vec{z}_{k-1}).\end{cases} \tag{2.11}$$

from the equation (2.5) and (2.6) we have

$$\begin{cases}\Re_k(\vec{x}_{k-1})=x_{k-1}(t)+\displaystyle\int_0^t \beta\left[\sum_{i=1}^{k-1} x_i(t)\,y_{k-1-i}(t)\right]dt-\big(x_0(t)-\chi_k x_0(t)\big),\\ \Re_k(\vec{y}_{k-1})=y_{k-1}(t)+\displaystyle\int_0^t \left[\beta\left(\sum_{i=1}^{k-1} x_i(t)\,y_{k-1-i}(t)\right)-\gamma y_{k-1}(t)\right]dt-\big(y_0(t)-\chi_k y_0(t)\big),\\ \Re_k(\vec{z}_{k-1})=z_{k-1}(t)+\displaystyle\int_0^t \gamma y_{k-1}(t)dt-(z_0(t)-\chi_k z_0(t)).\end{cases} \tag{2.12}$$

Considering equation (2.7), (2.13), (2.14) and the value of χ we can find the recursive expression.

$$\begin{cases} x_k = \chi_k x_{k-1}(t) + \hbar\left(x_{k-1}(t) + \int_0^t\left[\beta \sum_{i=0}^{k-1} x_i(t) y_{k-1-i}(t)dt - (x_0(t) - \chi_k(x_0(t))\right]\right), \\ y_k = \chi_k y_{k-1}(t) + \hbar(y_{k-1}(t) - \int_0^t \left[\beta \sum_{i=0}^{k-1} x_i(t) y_{k-1-i}(t) - y_{k-1}(t)\right]dt \\ \qquad\qquad\qquad\qquad -(y_0(t) - \chi_k\big(y_0(t)\big)), \\ z_k = \chi_k z_{k-1}(t) + \hbar\left(z_{k-1}(t) - \int_0^t \gamma\, y_{k-1}(t)dt - \big(z_0(t) - \chi_k(z_0(t))\big)\right). \end{cases}$$

Example

For numerical example here are the following values let: $N_1 = 20$, Initial population of $x(t)$, who are susceptible, $N_2 = 15$, Initial population of $y(t)$, who are infective, $N_3 = 10$, Initial population of $z(t)$, who are immune, $\beta = 0.01$, Rate of change of susceptible to infective population, $\gamma = 0.02$, Rate of change of infective to immune population.

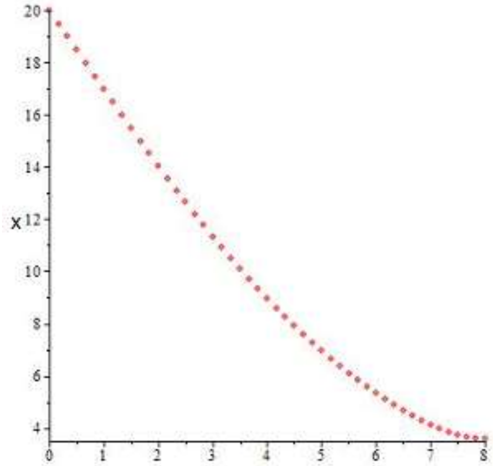

Fig 1

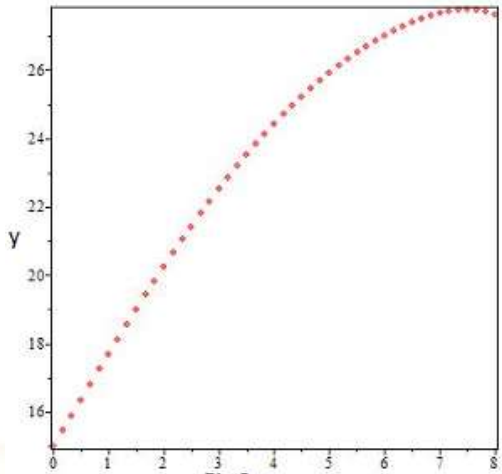

Fig 2

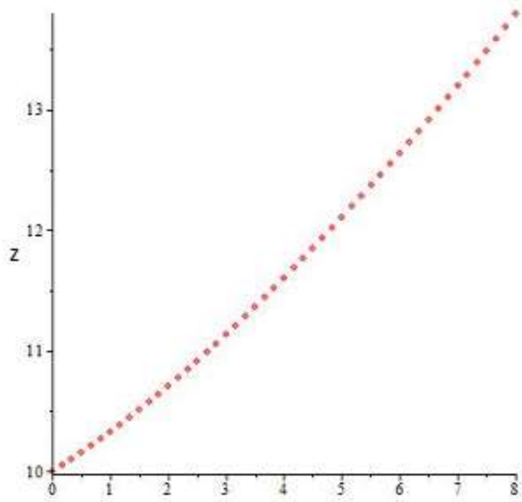

Fig 3

Fig describe the relation between X and t which continuously decreasing since the uninfected cell gradually decreasing and similarly the infected cell that is y gradually increasing and here z means who are immune to the disease that is also gradually increasing from time to time in fig 3.

CHAPTER 3- HIV-MODEL

Model → Uninfected cell die at rate an and uninfected cells with free virus produce the infected cell at a rate βnv.In equation (3.1) infected cell die at a rate bi and free virus is produced from infected cell at a rate ki and die at a rate δv. The variable z denotes the magnitude of the CTL (cytotoxic T lymphocyte) and CTL decay at a rate dz and infected cell are killed by CTL at rate ciz.Now consider the all the variable as a fuzzy variable and the fuzzy form of equation no (3.1). Let us consider the Novak may model in which I have included one more term,

$$\begin{cases} \frac{dn}{dt} = r - an - \beta nv \,, \\ \frac{di}{dt} = \beta nv - bi \,, \\ \frac{dv}{dt} = ki - \delta v \,, \\ \frac{dz}{dt} = ciz - dz \,, \end{cases} \tag{3.1}$$

where

$n \rightarrow uninfected\ cell,$

$i \rightarrow infected\ cell,$

$v \rightarrow no. of\ free\ virus\ particle,$

$z \rightarrow magnitude\ of\ the\ CTL(cytotoxic\ T\ lymphocyte)\ .$

Initial conditions of the fuzzy model.

$\tilde{n}$	(850,1000,1150)
$\tilde{\imath}$	(3,5,7)
$\tilde{v}$	(6750,7000,7250)
$\tilde{z}$	(1500,1300,1250)

Now fuzzy model form of the above equation will be,

$$\begin{cases} \frac{d\tilde{n}}{dt} = r - a\tilde{n} - \beta\tilde{n}\tilde{v}, \\ \frac{d\tilde{\imath}}{dt} = \beta\tilde{n}\tilde{v} - b\tilde{\imath}, \\ \frac{d\tilde{v}}{dt} = k\tilde{\imath} - \delta\tilde{v}, \\ \frac{d\tilde{z}}{dt} = c\tilde{\imath}\tilde{z} - d\tilde{z}. \end{cases} \tag{3.2}$$

Now consider the theorem (1.1) and let f be a fuzzy function where

$[f(x)]^{\alpha} = \left[\underline{f}^{\alpha}(x), \overline{f}^{\alpha}\right].$

So by using above previous theorem the system of fuzzy differentiable equations changes to the two crisp differential equation as follows,

$$\begin{cases} \dfrac{d\overline{n}^{\alpha}}{dt} = r - a\underline{n}^{\alpha} - \beta\underline{n}^{\alpha}\underline{v}^{\alpha}, \\ \dfrac{d\overline{i}^{\alpha}}{dt} = \beta\overline{n}^{\alpha}\overline{v}^{\alpha} - b\underline{i}^{\alpha}, \\ \dfrac{d\overline{v}^{\alpha}}{dt} = k\overline{i}^{\alpha} - \delta\underline{v}^{\alpha}, \\ \dfrac{d\overline{z}^{\alpha}}{dt} = c\overline{i}^{\alpha}\overline{z}^{\alpha} - d\underline{z}^{\alpha}. \end{cases} \qquad (3.2.a)$$

and

$$\begin{cases} \dfrac{d\underline{n}^{\alpha}}{dt} = r - a\overline{n}^{\alpha} - \beta\overline{n}^{\alpha}\overline{v}^{\alpha}, \\ \dfrac{d\underline{i}^{\alpha}}{dt} = \beta\underline{n}^{\alpha}\underline{v}^{\alpha} - b\overline{i}^{\alpha}, \\ \dfrac{d\underline{v}^{\alpha}}{dt} = k\underline{i}^{\alpha} - \delta\overline{v}^{\alpha}, \\ \dfrac{d\underline{z}^{\alpha}}{dt} = c\underline{i}^{\alpha}\underline{z}^{\alpha} - d\overline{z}^{\alpha}. \end{cases} \qquad (3.2.b)$$

Or

$$\begin{cases} \dfrac{d\overline{n}^{\alpha}}{dt} = r - a\overline{n}^{\alpha} - \beta\overline{n}^{\alpha}\overline{v}^{\alpha}, \\ \dfrac{d\overline{i}^{\alpha}}{dt} = \beta\underline{n}^{\alpha}\underline{v}^{\alpha} - b\overline{i}^{\alpha}, \\ \dfrac{d\overline{v}^{\alpha}}{dt} = k\underline{i}^{\alpha} - \delta\overline{v}^{\alpha}, \\ \dfrac{d\overline{z}^{\alpha}}{dt} = c\underline{i}^{\alpha}\underline{z}^{\alpha} - d\overline{z}^{\alpha}. \end{cases} \qquad (3.3.a)$$

and

$$\begin{cases} \dfrac{d\underline{n}^{\alpha}}{dt} = r - a\underline{n}^{\alpha} - \beta\underline{n}^{\alpha}\underline{v}^{\alpha}, \\ \dfrac{d\underline{i}^{\alpha}}{dt} = \beta\overline{n}^{\alpha}\overline{v}^{\alpha} - b\underline{i}^{\alpha}, \\ \dfrac{d\underline{v}^{\alpha}}{dt} = k\overline{i}^{\alpha} - \delta\underline{v}^{\alpha}, \\ \dfrac{d\underline{z}^{\alpha}}{dt} = c\overline{i}^{\alpha}\overline{z}^{\alpha} - d\underline{z}^{\alpha}. \end{cases} \qquad (3.3.b)$$

Now according to Zeroth order deformation equation

$(1-p)\mathcal{L}[\Phi(t;p) - v_0^0(t)] = p\hbar H(t)N[v(t)],$

and applying inverse operator i.e.

$$\frac{d(.)}{dt} = \int_0^t (.)\, dt,$$

then the equations $(3.3.a) and\ (3.3.b)$ will become,

$$\begin{cases} \overline{n}^{\alpha}(t) = \overline{n}^{\alpha}(0) + \int_0^t \left(r - a\underline{n}^{\alpha}(t) - \beta\underline{n}^{\alpha}(t)\underline{v}^{\alpha}(t)dt\right), \\ \overline{i}^{\alpha}(t) = \overline{i}^{\alpha}(0) + \int_0^t \left(\beta\underline{n}^{\alpha}(t)\underline{v}^{\alpha}(t) - b\overline{i}^{\alpha}(t)\right)dt. \\ \overline{v}^{\alpha}(t) = \overline{v}^{\alpha}(0) + \int_0^t \left(k\underline{i}^{\alpha}(t) - \delta\overline{v}^{\alpha}(t)\right)dt, \\ \overline{z}^{\alpha}(t) = \overline{z}^{\alpha}(0) + \int_0^t \left(c\underline{i}^{\alpha}(t)\underline{z}^{\alpha}(t) - d\overline{z}^{\alpha}(t)\right)dt. \end{cases} \quad (3.4.a)$$

And similarly for lower cut,

$$\begin{cases} \underline{n}^{\alpha}(t) = \underline{n}^{\alpha}(0) + \int_0^t \left(r - a\overline{n}^{\alpha}(t) - \beta\overline{n}^{\alpha}(t)\overline{v}^{\alpha}(t)\right)dt, \\ \underline{i}^{\alpha}(t) = \underline{i}^{\alpha}(0) + \int_0^t \left(\beta\overline{n}^{\alpha}(t)\overline{v}^{\alpha}(t) - b\underline{i}^{\alpha}(t)\right)dt, \\ \underline{v}^{\alpha}(t) = \underline{v}^{\alpha}(0) + \int_0^t \left(k\overline{i}^{\alpha}(t) - \delta\underline{v}^{\alpha}(t)\right)dt, \\ \underline{z}^{\alpha}(t) = \underline{z}^{\alpha}(0) + \int_0^t \left(c\overline{i}^{\alpha}(t)\overline{z}^{\alpha}(t) - d\underline{z}^{\alpha}(t)\right)dt. \end{cases} \quad (3.4.b)$$

Applying zeroth order deformation equation on above equation we will get the following equation in which $\mathcal{L}$ is the linear part and N is the nonlinear part

$$\begin{cases} (1-p)\left[\overline{n}^{\alpha}(t,p) - \overline{n}_0^{\alpha}(t)\right] = p\hbar\left(\overline{n}^{\alpha}(t,p) - \overline{n}^{\alpha}(0) - \int_0^t \left(r - a\underline{n}^{\alpha}(t) - \beta\underline{n}^{\alpha}(t)\underline{v}^{\alpha}(t)dt\right)\right), \\ (1-p)\left[\overline{i}^{\alpha}(t,p) - \overline{i}_0^{\alpha}(t)\right] = p\hbar\left(\overline{i}^{\alpha}(t,p) - \overline{i}^{\alpha}(0) - \int_0^t \left(\beta\underline{n}^{\alpha}(t)\underline{v}^{\alpha}(t) - b\overline{i}^{\alpha}(t)\right)dt\right), \\ (1-p)\left[\overline{v}^{\alpha}(t,p) - \overline{v}_0^{\alpha}(t)\right] = p\hbar\left(\overline{v}^{\alpha}(t,p) - \overline{v}^{\alpha}(0) - \int_0^t \left(k\underline{i}^{\alpha}(t) - \delta\overline{v}^{\alpha}(t)\right)dt\right), \\ (1-p)\left[\overline{z}^{\alpha}(t,p) - \overline{z}_0^{\alpha}(t)\right] = p\hbar\left(\overline{z}^{\alpha}(t,p) - \overline{z}^{\alpha}(0) - \int_0^t \left(c\underline{i}^{\alpha}(t)\underline{z}^{\alpha}(t) - d\overline{z}^{\alpha}(t)\right)dt\right). \end{cases}$$

$$(3.5.a)$$

$$\begin{cases} (1-p)\left[\underline{n}^{\alpha}(t,p) - \underline{n}_0^{\alpha}(t)\right] = p\hbar\left(\underline{n}^{\alpha}(t,p) - \underline{n}^{\alpha}(0) - \int_0^t \left(r - a\overline{n}^{\alpha}(t) - \beta\overline{n}^{\alpha}(t)\overline{v}^{\alpha}(t)\right)dt\right), \\ (1-p)\left[\underline{i}^{\alpha}(t,p) - \underline{i}_0^{\alpha}(t)\right] = p\hbar\left(\underline{i}^{\alpha}(t,p) - \underline{i}^{\alpha}(0) - \int_0^t \left(\beta\overline{n}^{\alpha}(t)\overline{v}^{\alpha}(t) - b\underline{i}^{\alpha}(t)\right)dt\right), \\ (1-p)\left[\underline{v}^{\alpha}(t,p) - \underline{v}_0^{\alpha}(t)\right] = p\hbar\left(\underline{v}^{\alpha}(t,p) - \underline{v}^{\alpha}(0) - \int_0^t \left(k\overline{i}^{\alpha}(t) - \delta\underline{v}^{\alpha}(t)\right)dt\right), \\ (1-p)\left[\underline{z}^{\alpha}(t,p) - \underline{z}_0^{\alpha}(t)\right] = p\hbar\left(\underline{z}^{\alpha}(t,p) - \underline{z}^{\alpha}(0) - \int_0^t \left(c\overline{i}^{\alpha}(t)\overline{z}^{\alpha}(t) - d\underline{z}^{\alpha}(t)\right)dt\right). \end{cases}$$

$$(3.5.b)$$

Substituting the value of $p = 0$ we will get the following result,

$$\begin{cases} \overline{n}^{\alpha}(t,0) = \overline{n}_0^{\alpha}(t), \\ \overline{i}^{\alpha}(t,0) = \overline{i}_0^{\alpha}(t), \\ \overline{v}^{\alpha}(t,0) = \overline{v}_0^{\alpha}(t), \\ \overline{z}^{\alpha}(t,0) = \overline{z}_0^{\alpha}(t). \end{cases} \quad (3.6.a)$$

and

$$\begin{cases} \underline{n}^{\alpha}(t,0) = \underline{n}_0^{\alpha}(t), \\ \underline{i}^{\alpha}(t,0) = \underline{i}_0^{\alpha}(t), \\ \underline{v}^{\alpha}(t,0) = \underline{v}_0^{\alpha}(t), \\ \underline{z}^{\alpha}(t,0) = \underline{z}_0^{\alpha}(t). \end{cases} \quad (3.6.b)$$

Now substituting the value of $p = 1$ we will get

$$\begin{cases} \overline{n}^{\alpha}(t,1) - \overline{n}^{\alpha}(t), \\ \overline{i}^{\alpha}(t,1) - \overline{i}^{\alpha}(t), \\ \overline{v}^{\alpha}(t,1) - \overline{v}^{\alpha}(t), \\ \overline{z}^{\alpha}(t,1) - \overline{z}^{\alpha}(t). \end{cases} \quad (3.7.a)$$

and

$$\begin{cases} \underline{n}^{\alpha}(t,1) - \underline{n}^{\alpha}(t), \\ \underline{i}^{\alpha}(t,1) - \underline{i}^{\alpha}(t), \\ \underline{v}^{\alpha}(t,1) - \underline{v}^{\alpha}(t), \\ \underline{z}^{\alpha}(t,1) - \underline{z}^{\alpha}(t). \end{cases} \quad (3.7.b)$$

Now we know the Taylor series and apply with respect to q

$$Q(t,q) = x_0(t) + \sum_{m=1}^{\infty} x_m(t) q^m ,$$

$$x_m(t) = \frac{\partial^m Q(t,q)}{\partial q^m} \quad at\ q = 0,$$

and according to above Taylor equation we will apply in equation (3.7.a) and (3.7.b) we will get following equation

$$\begin{cases} \overline{n}^{\alpha}(t,p) = \overline{n}_0^{\alpha}(t) + \sum \overline{n}_k^{\alpha}(t) p^k, \\ \overline{i}^{\alpha}(t,1) = \overline{i}_0^{\alpha}(t) + \sum \overline{i}_k^{\alpha}(t) p^k , \\ \overline{v}^{\alpha}(t,1) = \overline{v}_0^{\alpha}(t) + \sum \overline{v}_k^{\alpha}(t) p^k, \\ \overline{z}^{\alpha}(t,1) = \overline{z}_0^{\alpha}(t) + \sum \overline{z}_k^{\alpha}(t) p^k . \end{cases} \quad (3.8.a)$$

and the other equation will be

$$\begin{cases} \underline{n}^{\alpha}(t,1) = \underline{n}_0^{\alpha}(t) + \sum \underline{n}_k^{\alpha}(t)p^k, \\ \underline{i}^{\alpha}(t,1) = \underline{i}_0^{\alpha}(t) + \sum \underline{i}_k^{\alpha}(t)p^k\,, \\ \underline{v}^{\alpha}(t,1) = \underline{v}_0^{\alpha}(t) + \sum \underline{v}_k^{\alpha}(t)p^k, \\ \underline{z}^{\alpha}(t,1) = \underline{z}_0^{\alpha}(t) + \sum \underline{z}_k^{\alpha}(t)p^k\,, \end{cases} \quad (3.8.b)$$

On the basis of Taylor equation equations (3.8.a) and (3.8.b) will become

$$\begin{cases} \overline{n}_0^{\alpha k}(t) = \dfrac{\partial^k \overline{n}^{\alpha}(t,p)}{\partial p^k}, \\ \overline{i}_0^{\alpha k}(t) = \dfrac{\partial^k \overline{i}^{\alpha}(t,p)}{\partial p^k}, \\ \overline{v}_0^{\alpha k}(t) = \dfrac{\partial^k \overline{v}^{\alpha}(t,p)}{\partial p^k}, \\ \overline{z}_0^{\alpha k}(t) = \dfrac{\partial^k \overline{z}^{\alpha}(t,p)}{\partial p^k}. \end{cases} \quad (3.9.a)$$

and

$$\begin{cases} \underline{n}_0^{\alpha k}(t) = \dfrac{\partial^k \underline{n}^{\alpha}(t,p)}{\partial p^k}, \\ \underline{i}_0^{\alpha k}(t) = \dfrac{\partial^k \underline{n}^{\alpha}(t,p)}{\partial p^k}, \\ \underline{v}_0^{\alpha k}(t) = \dfrac{\partial^k \underline{n}^{\alpha}(t,p)}{\partial p^k}, \\ \underline{z}_0^{\alpha k}(t) = \dfrac{\partial^k \underline{n}^{\alpha}(t,p)}{\partial p^k}. \end{cases} \quad (3.9.b)$$

from equation of (3.9.a) and (3.9.b) we will get

$$\begin{cases} \overline{n}_k^{\alpha}(t) = \overline{n}_0^{\alpha k}(t), \\ \overline{i}_k^{\alpha}(t) = \overline{i}_0^{\alpha k}(t), \\ \overline{v}_k^{\alpha}(t) = \overline{v}_0^{\alpha k}(t), \\ \overline{z}_k^{\alpha}(t) = \overline{z}_0^{\alpha k}(t). \end{cases} \quad (3.10.a)$$

and

$$\begin{cases} \underline{n}_k^{\alpha}(t) = \underline{n}_0^{\alpha k}(t), \\ \underline{i}_k^{\alpha}(t) = \underline{i}_0^{\alpha k}(t), \\ \underline{v}_k^{\alpha}(t) = \underline{v}_0^{\alpha k}(t), \\ \underline{z}_k^{\alpha}(t) = \underline{z}_0^{\alpha k}(t). \end{cases} \quad (3.10.b)$$

Using p=1 in equation in (8.1) and (8.b) we will get the following equation

$$\begin{cases} \overline{n}^{\alpha}(t) = \overline{n}_0^{\alpha}(t) + \sum_{k=1}^{\infty} \overline{n}_k^{\alpha}(t), \\ \overline{i}^{\alpha}(t) = \overline{n}_0^{\alpha}(t) + \sum_{k=1}^{\infty} \overline{i}_k^{\alpha}(t), \\ \overline{v}^{\alpha}(t) = \overline{n}_0^{\alpha}(t) + \displaystyle\sum_{k=1}^{\infty} \overline{v}_k^{\alpha}(t), \\ \overline{z}^{\alpha}(t) = \overline{n}_0^{\alpha}(t) + \displaystyle\sum_{k=1}^{\infty} \overline{z}_k^{\alpha}(t), \end{cases} \quad (3.11.a)$$

and

$$\begin{cases} \underline{n}^{\alpha}(t) = \underline{n}_0^{\alpha}(t) + \sum_{k=1}^{\infty} \underline{n}_k^{\alpha}(t), \\ \underline{i}^{\alpha}(t) = \underline{i}_0^{\alpha}(t) + \sum_{k=1}^{\infty} \underline{n}_k^{\alpha}(t), \\ \underline{v}^{\alpha}(t,1) = \underline{v}_0^{\alpha}(t) + \displaystyle\sum_{k=1}^{\infty} \underline{n}_k^{\alpha}(t), \\ \underline{z}^{\alpha}(t,1) = \underline{z}_0^{\alpha}(t) + \displaystyle\sum_{k=1}^{\infty} \underline{n}_k^{\alpha}(t). \end{cases} \quad (3.11.b)$$

Vector Representation

Define the vector for the upper cut,

$$\begin{aligned} \vec{\overline{n}}_k^{\alpha} &= \{\overline{n}_0^{\alpha}(t), \overline{n}_1^{\alpha}(t), \overline{n}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots \dots \overline{n}_k^{\alpha}(t)\}, \\ \vec{\overline{i}}_k^{\alpha} &= \left\{\overline{i}_0^{\alpha}(t), \overline{i}_1^{\alpha}(t), \overline{i}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots \dots \overline{i}_k^{\alpha}(t)\right\}, \\ \vec{\overline{v}}_k^{\alpha} &= \{\overline{v}_0^{\alpha}(t), \overline{v}_1^{\alpha}(t), \overline{v}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots \dots \overline{v}_k^{\alpha}(t)\}, \\ \vec{\overline{z}}_k^{\alpha} &= \{\overline{z}_0^{\alpha}(t), \overline{z}_1^{\alpha}(t), \overline{z}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots \dots \overline{z}_k^{\alpha}(t)\}, \end{aligned} \quad (3.12.a)$$

and for lower cut

$$\begin{aligned} \vec{\underline{n}}_k^{\alpha} &= \{\underline{n}_0^{\alpha}(t), \underline{n}_1^{\alpha}(t)\underline{n}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots .. \underline{n}_0^{\alpha}(t)\}, \\ \vec{\underline{i}}_k^{\alpha} &= \{\underline{i}_0^{\alpha}(t), \underline{i}_1^{\alpha}(t)\underline{i}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots \dots \dots \underline{i}_0^{\alpha}(t)\}, \\ \vec{\underline{v}}_k^{\alpha} &= \{\underline{v}_0^{\alpha}(t), \underline{v}_1^{\alpha}(t)\underline{v}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots .. \underline{v}_0^{\alpha}(t)\}, \\ \vec{\underline{z}}_k^{\alpha} &= \{\underline{z}_0^{\alpha}(t), \underline{z}_1^{\alpha}(t)\underline{z}_2^{\alpha}(t) \dots \dots \dots \dots \dots \dots ... \underline{z}_0^{\alpha}(t)\}. \end{aligned} \quad (3.12.b)$$

k^{th} deformation equation for upper cut will be

$$\begin{aligned} &\mathcal{L}[\overline{n}_k^{\alpha}(t) - \chi_k \overline{n}_{k-1}^{\alpha}(t) = \hbar \Re_{\overline{n}_k}(\overline{n}_{k-1}^{\alpha})], \\ &\mathcal{L}\left[\overline{i}_k^{\alpha}(t) - \chi_k \overline{i}_{k-1}^{\alpha}(t) = \hbar \Re_{\overline{n}_k}(\overline{i}_{k-1}^{\alpha})\right], \\ &\mathcal{L}[\overline{v}_k^{\alpha}(t) - \chi_k \overline{v}_{k-1}^{\alpha}(t) = \hbar \Re_{\overline{n}_k}(\overline{v}_{k-1}^{\alpha})], \\ &\mathcal{L}[\overline{z}_k^{\alpha}(t) - \chi_k \overline{z}_{k-1}^{\alpha}(t) = \hbar \Re_{\overline{n}_k}(\overline{z}_{k-1}^{\alpha})], \end{aligned} \quad (3.13.a)$$

and k^{th} deformation equation for lower cut will be

$$\mathcal{L}[\underline{n}_k^\alpha(t) - \chi_k \underline{n}_{k-1}^\alpha(t) = \hbar \Re_{\overline{n}_k}(\underline{n}_{k-1}^\alpha)],$$
$$\mathcal{L}[\underline{i}_k^\alpha(t) - \chi_k \underline{i}_{k-1}^\alpha(t) = \hbar \Re_{\overline{n}_k}(\underline{i}_{k-1}^\alpha)],$$
$$\mathcal{L}[\underline{v}_k^\alpha(t) - \chi_k \underline{v}_{k-1}^\alpha(t) = \hbar \Re_{\overline{n}_k}(\underline{v}_{k-1}^\alpha)], \tag{3.13.b}$$
$$\mathcal{L}[\underline{z}_k^\alpha(t) - \chi_k \underline{z}_{k-1}^\alpha(t) = \hbar \Re_{\overline{n}_k}(\underline{z}_{k-1}^\alpha)].$$

Table 1. Parameters of the HIV model

$r = 7$	$a = 0.007$	$\beta = 42163 * 10^{11}$
$b = 0.0999$	$s = 0.2$	$k = 90.67$

Solution

Our final equation for upper cut will be

$$\overline{n}_k^\alpha(t) = \chi_k \overline{n}_{k-1}(t) + \hbar\left(\overline{n}_{k-1}(t)\right) + \int_0^t \left(\sum_{i=1}^{k-1} \left(r - a\underline{n}_i^\alpha(t) - \beta a \underline{n}_i^\alpha(t) \underline{v}_{k-2i-1}^\alpha(t)\right)\right) dt - \left(\overline{n}_0^\alpha(t) - \chi_k \overline{n}_0^\alpha(t)\right),$$

$$\overline{i}_k^\alpha(t) = \chi_k \overline{i}_{k-1}(t) + \hbar\left(\overline{i}_{k-1}(t)\right) + \int_0^t \left(\sum_{i=1}^{k-1} \left(\beta \underline{n}_i^\alpha(t) \underline{v}_{k-2i-1}^\alpha(t)\right) - b\overline{i}_{k-1-i}^\alpha\right) dt - \left(\overline{i}_0^\alpha(t) - \chi_k i_0^\alpha(t)\right),$$

$$\overline{v}_k^\alpha(t) = \chi_k \overline{v}_{k-1}^\alpha(t) + \hbar \overline{v}_{k-1}^\alpha(t) + \int_0^t \left(\sum_{i=1}^{k-1} \left(k\underline{i}_{k-1-i}^\alpha(t) - \delta \overline{v}_{k-1-2i}^\alpha(t)\right)\right) dt - \left(\overline{v}_0^\alpha(t) - \chi_k \overline{v}_0^\alpha(t)\right),$$

$$\overline{z}_k^\alpha(t) = \chi_k \overline{z}_{k-1}^\alpha(t) + \hbar \overline{z}_{k-1}^\alpha(t) + \int_0^t \left(\sum_{i=1}^{k-1} \left(c\underline{i}_{k-1-i}^\alpha(t) \underline{z}_{k-1-3i}^\alpha(t) - d\overline{z}_{k-1-3i}^\alpha(t)\right)\right) dt - \left(\overline{z}_0^\alpha(t) - \chi_k \overline{z}_0^\alpha(t)\right).$$

Table 4. α-cut of the initial conditions,

$\overline{n}^0(0)$	$1000\alpha + 1150(1-\alpha)$
$\underline{n}^0(0)$	$1000\alpha + 850(1-\alpha)$
$\overline{i}^0(0)$	$5\alpha + 6(1-\alpha)$
$\underline{i}^0(0)$	$5\alpha + 4(1-\alpha)$

$\overline{v}^0(0)$	$7000\alpha + 7250(1-\alpha)$
$\underline{v}^0(0)$	$7000\alpha + 6750(1-\alpha)$
$\overline{z}^0(0)$	$1500\alpha + 1300(1-\alpha)$
$\underline{z}^0(0)$	$1500\alpha + 1250(1-\alpha)$

Final solution for lower cut

$$\underline{n}_k^\alpha(t) = \chi_k \underline{n}_{k-1}(t) + \hbar\left(\underline{n}_{k-1}(t)\right) + \int_0^t \left(\sum_{i=1}^{k-1}\left(r - a\overline{n}_i^\alpha(t) - \beta a\overline{n}_i^\alpha(t)\overline{v}_{k-2i-1}^\alpha(t)\right)\right)dt - \left(\underline{n}_0^\alpha(t) - \chi_k\underline{n}_0^\alpha(t)\right),$$

$$\underline{i}_k^\alpha(t) = \chi_k \underline{i}_{k-1}(t) + \hbar\left(\underline{i}_{k-1}(t)\right) + \int_0^t \left(\sum_{i=1}^{k-1}\left(\beta\overline{n}_i^\alpha(t)\overline{v}_{k-2i-1}^\alpha(t)\right) - b\underline{i}_{k-1-i}^\alpha\right)dt - \left(\underline{i}_0^\alpha(t) - \chi_k\underline{i}_0^\alpha(t)\right),$$

$$\underline{v}_k^\alpha(t) = \chi_k \underline{v}_{k-1}^\alpha(t) + \hbar\underline{v}_{k-1}^\alpha(t) + \int_0^t \left(\sum_{i=1}^{k-1}\left(k\overline{i}_{k-1-i}^\alpha(t) - \delta\underline{v}_{k-1-2i}^\alpha(t)\right)\right)dt - \left(\underline{v}_0^\alpha(t) - \chi_k\underline{v}_0^\alpha(t)\right),$$

$$\underline{z}_k^\alpha(t) = \chi_k \underline{z}_{k-1}^\alpha(t) + \hbar\underline{z}_{k-1}^\alpha(t) + \int_0^t \left(\sum_{i=1}^{k-1}\left(c\overline{i}_{k-1-i}^\alpha(t)\overline{z}_{k-1-3i}^\alpha(t) - d\underline{z}_{k-1-3i}^\alpha(t)\right)\right)dt - \left(\underline{z}_0^\alpha(t) - \chi_k\underline{z}_0^\alpha(t)\right).$$

Graph

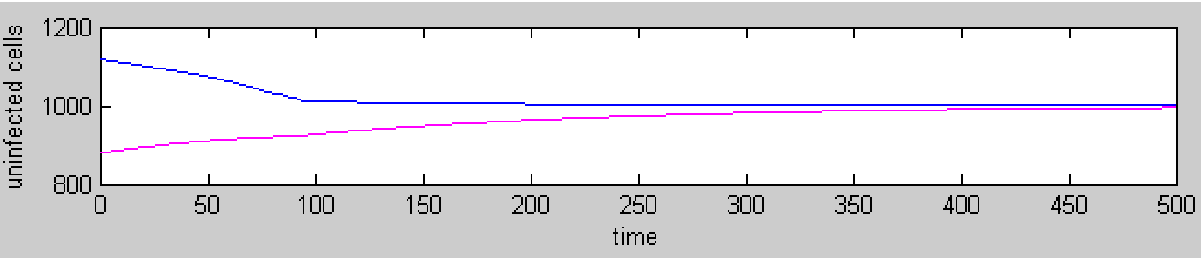

Fig 1

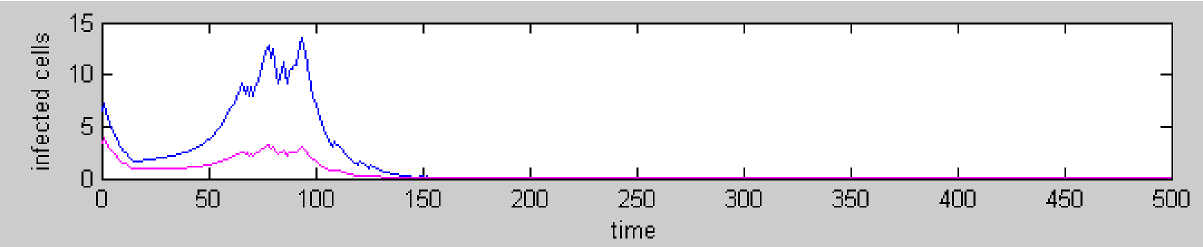

Fig 2

Fig 1 and Fig 2 describe the relation between uninfected cell and infected cell with time .In fig1 describe the above bound and lower bound of the function which gradually increasing but in Fig 2 infected cell increasing with the time .

Conclusion – Homotopy analysis method is known to be a powerful method for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this thesis, we used homotopy analysis method for solving a system of differential equation using Zeroth order deformation equation which play a very important role in relation between nonlinear equation to linear equation that are describe in SIR model for an epidemic disease and HIV model. Using the basic fundamental of ħ curve, we are able to find the area of convergence in the series solution. Numerical example also provided to show the simplicity and efficiency of the method.

References

[1] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, **24**, 319-330 (1987).

[2] M. Najariyan, M. H. Farahi, and M. Alavian / TJMCS Vol .2 No.4 (2011) 639-649

[3] J. J. Buckley, T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems, **110**, 43-54 (2000).

[4] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Methods*, Chapman and Hall/CRC Press, Boca Raton, (2003).

[5] S. J. Liao, On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computationm, **147**, 499-513 (2004).

[6] Z. Odibat, S. Momani, H. Xu, A reliable algorithm of Homotopy analysis method for solving nonlinear fractional differential equations, Applied Mathematical Modelling, **34**, 593-600 (2010).

[7] Nowak, M. A., and May, R. M., "Mathematical biology of HIV infections, Antigenic variation and diversity threshold", Mathematical Biosciences, 106(1) pp. 1-21, 1991.

[8] Nowak, M.A., and May, R. M., "*Virus dynamic*", Oxford University press, 2000.

[9] Nieto, J.J., Khastan, A., and Ivaz.K., ”Numerical solution of fuzzy differential equations under generalized differentiability”, Nonlinear Analysis: Hybrid Systems 3, pp.700-707, 2009

[10] Omar Abu Arqub ,Ahmad El Ajou ,and Nabil Shawagfeh Appl. Math. Inf. Sci.7, No. 5,1903-1919(2013)

[11] Vahdati S. ,Tavassoli Kajani M. and Ghasemi M. ISSN 2277-2502
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