METHODS TO DETECT CHAOS AND BIFURCATION ANALYSIS OF NONLINEAR SYSTEMS

THESIS FOR THE DEGREE OF

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UNDER THE GUIDANCE OF

PROF. BIPLAB GANGULI

BY

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CERTIFICATE

This is to certify that the project thesis entitled 'METHODS TO DETECT CHAOS AND BIFURCATION ANALYSIS OF SOME NONLINEAR SYSTEMS' submitted by ABHISEK BAG, a 2 year M.Sc. student of Department of Physics and Astronomy, National Institute of Technology Rourkela is partial fulfilment for the award of the degree of Master of Science in Physics and has been carried out by him under my supervision and guidance. The results incorporated in the thesis have been reproduced.

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ABSTRACT

Chaos is a recent but well-established phenomenon. In this report work has been done in the period of July,2014 to April,2015.The main work done in this paper is to study various techniques to find whether any system is chaotic or not. We have tested these systems chaotic behavior with respect to different parameter values. Work basically is done using MATLAB programming. The Matlab codes have been used is given in the 'Matlab codes' section. Mathieu Oscillator and Logistic equation are used as systems to study.

CONTENTS

Chaos theory

Chaos basically means disorder. Change and time are the basic building block of chaos theory. A system is an assemble of interacting parts. Now system changes with time. Such as the weather, market prices anything can be a system. Now what will be the qualitative long-term behavior of a changing system. The presence of chaos means the long-term prediction is worthless. It generally represents irregular behavior over time. The value at present largely depends on the previous value. Chaos may not appear easy but its origin may be very simple. I have given a very simple equation to show how a simple equation can show a very complex behavior.

$$X_{n+1} = 2.5 - X_n^2$$

Input Value(X_n)	New Value(X_{n+1})
1	1.5
1.5	0.25
0.25	2.43

Etc

This gives a widely fluctuating values. Thus, a chaotic system looks unarranged but it is deterministic. The above equation is quite simple but its behavior is not. Chaotic systems carry out certain laws or equations. The chaotic behavior can come out of a single variable. Now a system can alter with time or distance. Depending on that chaos is categorised in two broad class one is temporal chaos and another is spatial chaos. Chaos is seen only in deterministic non-linear dynamical systems.

Nonlinear is something which is not linear. The simple example of a nonlinear system is freezing of water. Up to zero degree, nothing occurs but unexpectedly at zero degree the water freezes to ice. To study the nonlinear systems we have to inspect three things

● long-term behavior

●response to small stimuli

●persistence of local pulses

A dynamical system is something which changes with time. There are two types of dynamical systems one is conservative where no energy is lost and another is dissipative where the system has friction, the system releases energy and comes near to limiting condition .chaos occurs in these cases. Chaos theory is a multidisciplinary subject. It is seen in mathematics, physics, biology, chemistry, geology, medicine, physiology, ecology, atmospheric science, oceanology, astronomy, solar science etc. The main reasons for chaos to occur are these

● An increase in the control factor to a high value. Even if normal ecosystem does not go chaotic but they can if human interfere.

● The nonlinear interaction of two or more systems can cause chaos. Single pendulum is not chaotic but double pendulum can show chaotic behavior

● The aftermath of environmental noise can cause chaos to occur.

Mathieu equation

Parametric oscillators are those in which we can get oscillation just changing some parameter values. Just like while riding a swing we can control the oscillation just by standing up and sitting down at particular positions of the oscillation. Oscillator is a differential equation which have bounded solution. Being periodic is not necessary. We will be looking for aperiodic motion. We can derive Mathieu equation by considering a pendulum with a thin rod oscillates in a vertical motion

$$X'' = -\omega_0^2(1 + \mathrm{a}\cos(\omega t))sinX$$

Considering friction the real Mathieu equation will be

$$X'' = -2\Delta X' - \omega_0^2(1 + \mathrm{a}\cos(\omega t))sinX$$

This equation can be approximated to many physical systems. Thus studying this equation can be real helpful to study many real life phenomenon. That's why we have considered the friction also.

The equation above is an example of a non-autonomous system. We make it autonomous by proposing Z= ωt. So equation can be reconstructed as

$$X' = Y$$
$$Y' = -2\Delta Y - \omega_0^2(1 + a\cos(Z))sinX$$
$$Z' = \omega$$

Mathematically we can treat Z as a new variable although the equation physically still is non-autonomous. Now there are four parameters$(\Delta, \omega_0, \omega, a)$.

Among them, only a and ω controls the excitation. We have tried to see whether the system is chaotic for what 'a' values .The rest of the parameters are kept constant.

$$\omega_0 = 1, \Delta = 0.5, \omega = 2$$

Mathieu equation is a very complex equation with all these cosine, sine terms. So to make things less scary we have chosen these three parameter values constant for all the calculations. Now we will try to use the techniques described above to find chaos.

METHODS FOR DETECTION OF CHAOS

Chaos is a well-established phenomenon but characterizing the chaos is still under scrutiny. There are several techniques that we use to find whether any system is chaotic or not. We have used three techniques here.

●Lyapunov Exponents

●Dense Filled phase space

●Poincare Section

We will use a particular system called Mathieu Oscillator. Using above approaches, we will shot to find if there is chaos for any parameter value.

LYAPUNOV EXPONENTS

Lyapunov exponent is the most useful technique to characterize chaos. If we consider two close trajectories in the phase space then we see that in the radial direction they converge with time, while in the direction along the cycle they neither converge nor diverge. The convergence/divergence properties of nearby trajectories are characterised by the Lyapunov exponents. Suppose we start with a discrete dynamical system

$$X_{n+1} = f(X_n)$$

The Lyapunov exponents can be constructed as

$$b(X_0) = \frac{lim}{N \to \infty} \frac{lim}{\in \to 0} \left(\frac{1}{N} log \frac{f^N(X_0 + \in) - f^N(X_0)}{\in} \right)$$

$$= \frac{lim}{N \to \infty} \left(\frac{1}{N} log \frac{df^N(X_0)}{dX_0} \right)$$

We can say two trajectories in phase space with a separation of $|\delta X_0|$at time t=0, will have a separation of $|\delta X_t|$at t=t given by

$$|\delta X_t| = e^{bt} |\delta X_0|$$

Since, the rate of divergence of trajectories can be different in different directions in the phase space, the number of Lyapunov exponents is equal to the dimensionality of phase space.So we want to calculate the maximal Lyapunov exponent (MLE). If the phase space is solid, a positive MLE would mean that the prognosis of the system is impossible. The system is then labelled to be delicately dependent on initial conditions or chaotic.

The Lyapunov exponents represent the expanding and contracting nature of the different directions in phase space. The sum of Lyapunov exponents is negative for dissipative systems. Whereas positive Lyapunov means nearby trajectories are exponentially diverging. For a bound phase space if we get a positive LE that means we have chaos.

First we have calculated the LE for simple logistic equation just to understand how things work. The equation is

$$X_{t+1}' = r * X_t(1 - X_t)$$

This simply gives the information about population growth. Where r is Malthusian constant. X_t is the population at time t and X_t' is the population growth with respect to time. This equation was first discovered to study the population growth of bacteria. But this logistic equation is very powerful. Apart from its application in population growth it can also be applied to neural networks, medicine, economics and to many other streams.

Here (Fig1) we can see the Lyapunov exponents remain negative for $r<3.6$. At $r=3.6$, it becomes zero. It indicated the period doubling and appearance of chaos. For $r>3.6$ period doubling sequentially happens. This indicates strongly that there is chaos.

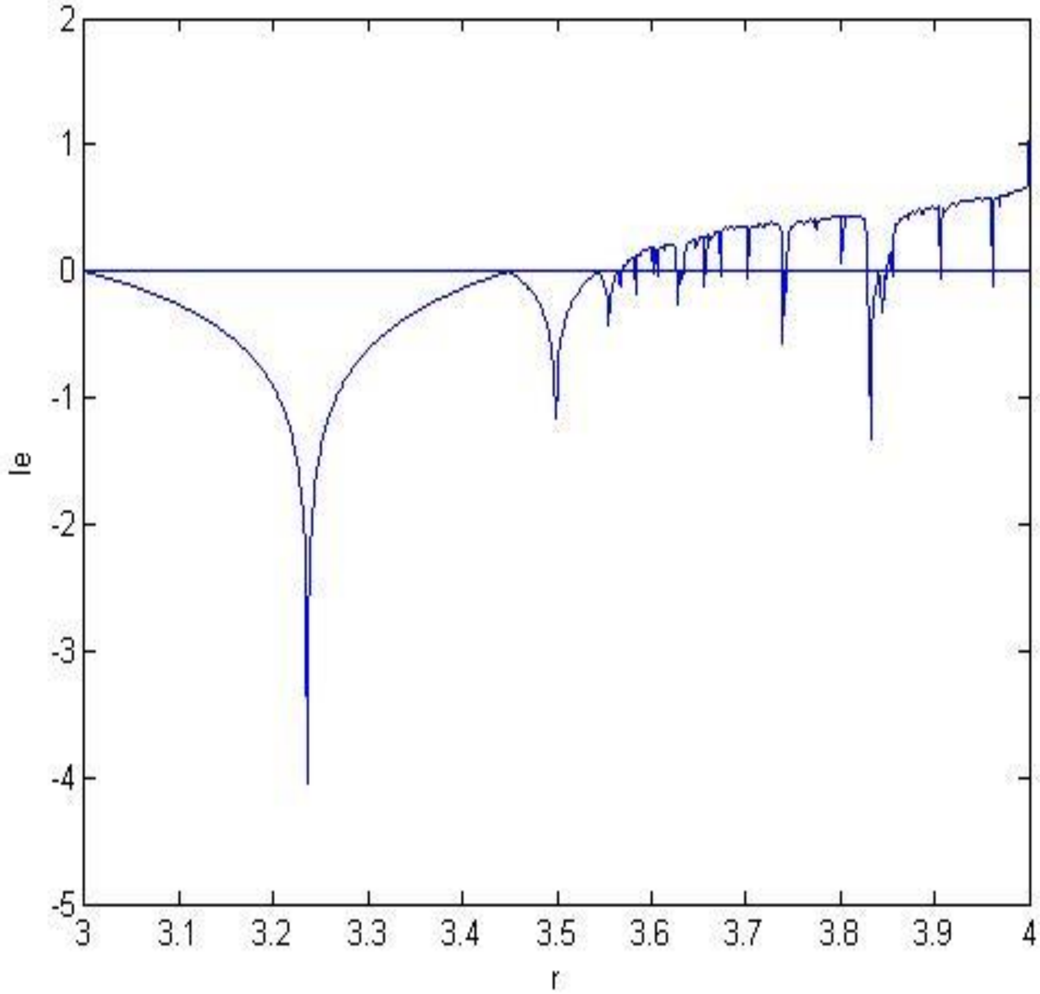

Fig 1.LE for logistic equation.r>3.6 can be chaotic

Now using the algorithm which is given in [2] we have calculated Lyapunov exponents for Mathieu equation. The graph has been plotted between Maximal Lyapunov Exponent (MLE) and system parameter a.

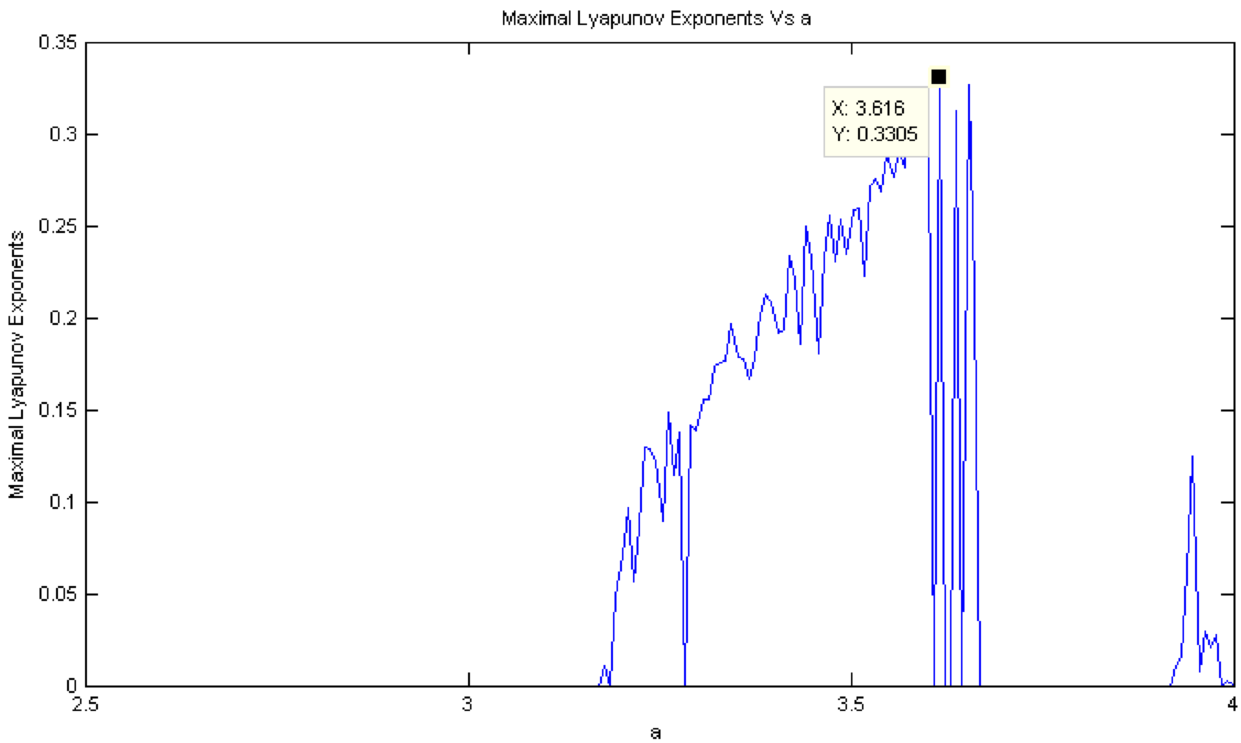

Fig2: The initial points for all the runs is [0,0.2,2].Each of LEs are taken after 200units of run

Here (Fig 2) we can see that we get MLE for a>3.2. Hence we should get chaos only above this a value whereas for a value less than 3.2 we should get a limit cycle if we plot the phase space diagram. Here the value of 'a' for which we get the highest MLE is 3.6.

Now we plot the phase space for a<3.2. We get a limit cycle for a=2.5. (Fig 3). Limit cycles are crutial phenomenon. This occurs only in non-linear systems. A limit cycle can be stable, unstable and half stable. This is a stable limit cycle because trajectories are converging towards each other. That means it can have self-sustained oscillation for each value of 'a' up to 3.2. This means energy dissipated in one cycle will balance the energy fill in another cycle. Thus the oscillation will continue without any external force. Later we will plot the Poincare section for a=2.7 and see if we really get a limit cycle or not. If the Poincare section only produces two points it would mean that we had a limit cycle.

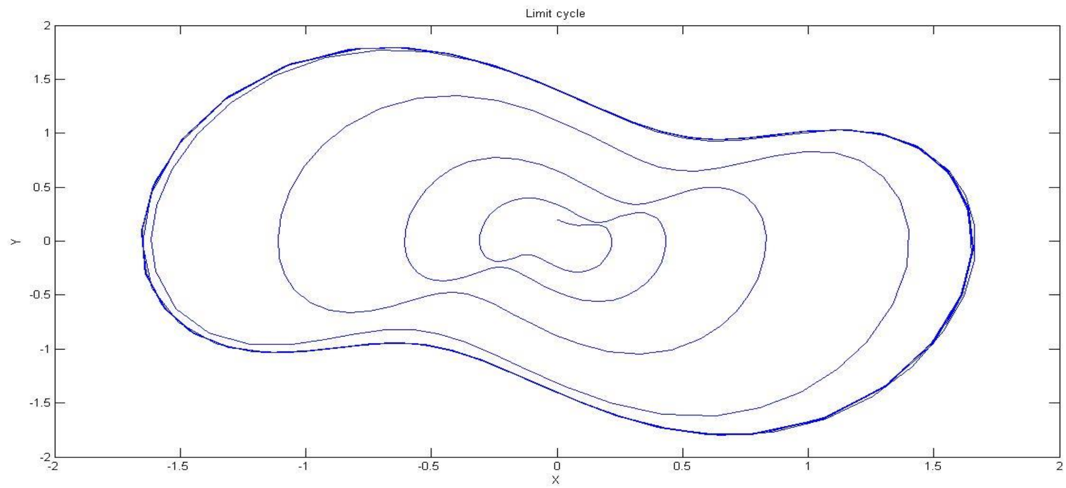

Fig3:The initial point for run [0,0.2,2].Run for 2000 units of time

Dense Filled Phase Space

Another diagnostic tool to detect chaos is dense filled phase space. This is called ergodic theory. If we plot the phase space diagram of a system and we get compact graphs for large time run then we can say we have deterministic dynamics. We have plotted here the phase space diagram of Mathieu equation.

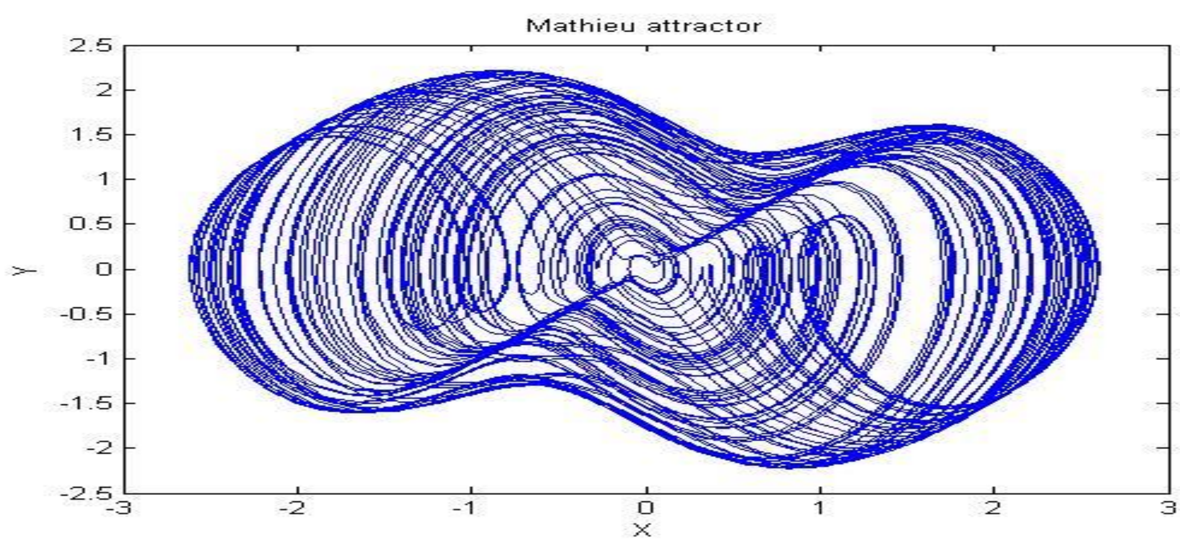

Fig4:Trajectory starting at [0,0.2,2] with a=3.5,Time simulation 500 units

Firstly solving the equations we get an attractor. Then we can see the transition from fig4 to fig5 how densely the phase space fills for larger time run.

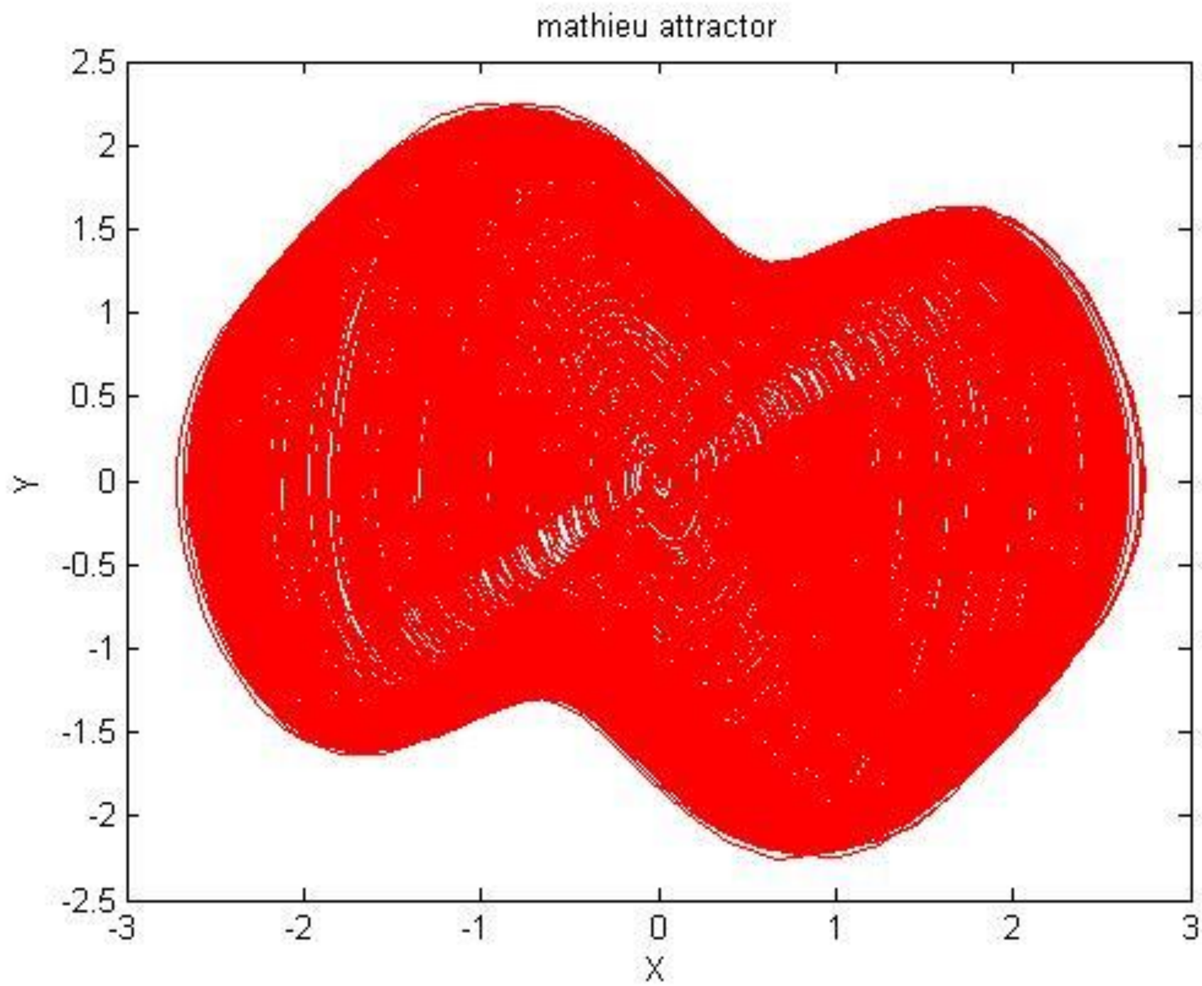

Fig5:Theintial point for run is [0,0.2,2]. a=3.5 Run for 5000 units of time

We can conclude that we have chaos for a=3.5.

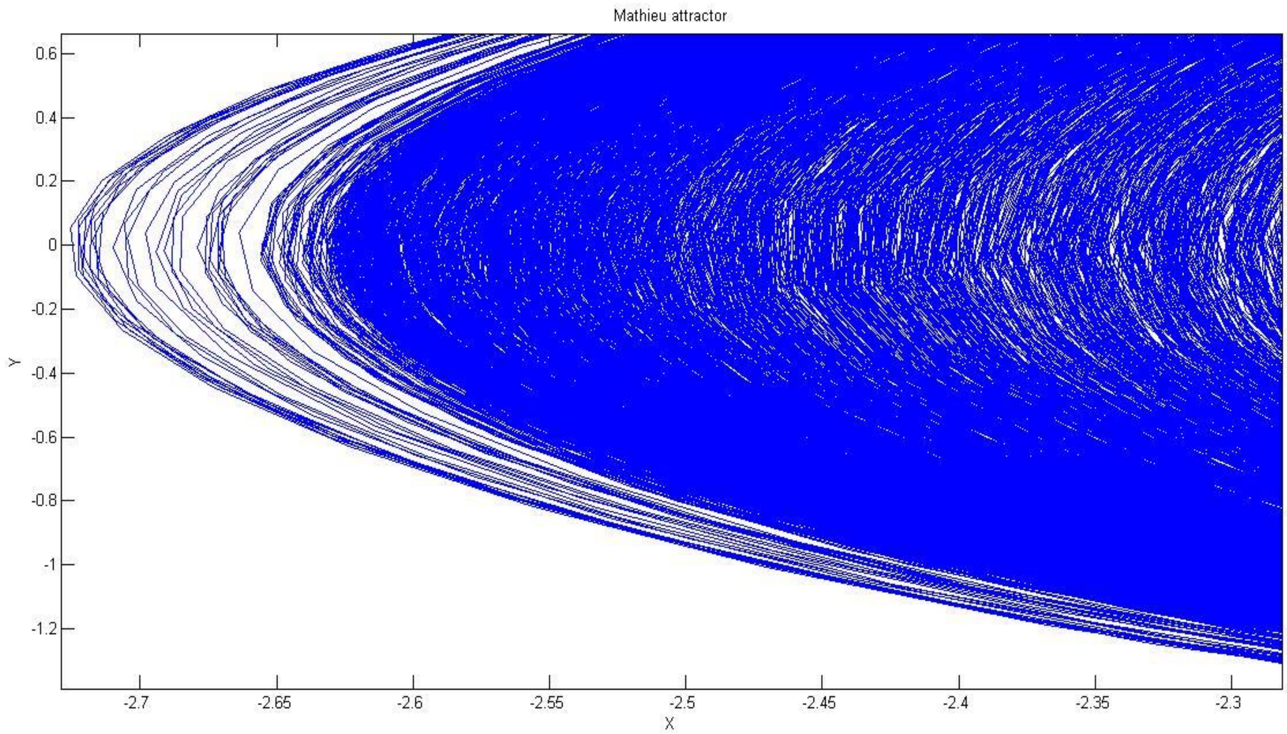

Fig6:Theintial point for run is [0,0.2,2].a=3.5 Run for 50000 units of time

POINCARE MAP

We treat higher dimensional autonomous nonlinear dynamical systems with Poincare section. Autonomous systems are those for which potential is time independent. To apply Poincare section we need

●3D phase space

●A plane in phase space through which orbits repeatedly pass

●An array of solutions for particle orbits

Poincare section reduces the dimensionality of phase space. That's why we need minimum 3D phase space. Chaos occurs in 3D phase space .Suppose we have trajectories in three dimension then if we want to take the Poincare section we will take a plane which will cut through the path and this plane will be called Poincare section. So it basically is the intersection of orbits in a surface. In Poincare section we use a trigger to note down the coordinates of the path. This trigger may be vanishing of one coordinate or a hyperspace cutting through the trajectories.

Let M be a (n-1) dimensional surface which is transverse to the trajectories of any dynamical systems.X_0 be a point on the surface M at time t=0. The trajectory will come to that surface again at X_1 at any later time $t=t_1$. Now if we define a mapping S from M to M.

$X_1=S(X_0)$

And after n+1 iterations

$X_{n+1}=S(X_n)$

Then map S is called Poincare map.

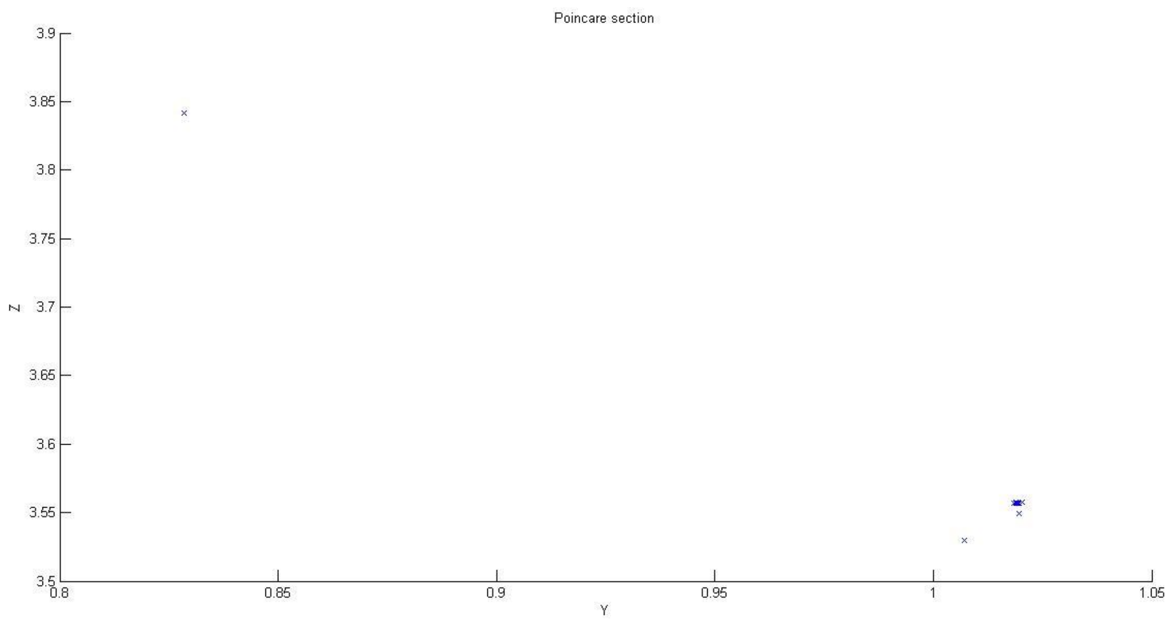

Fig7:Poincare section for a=2.7

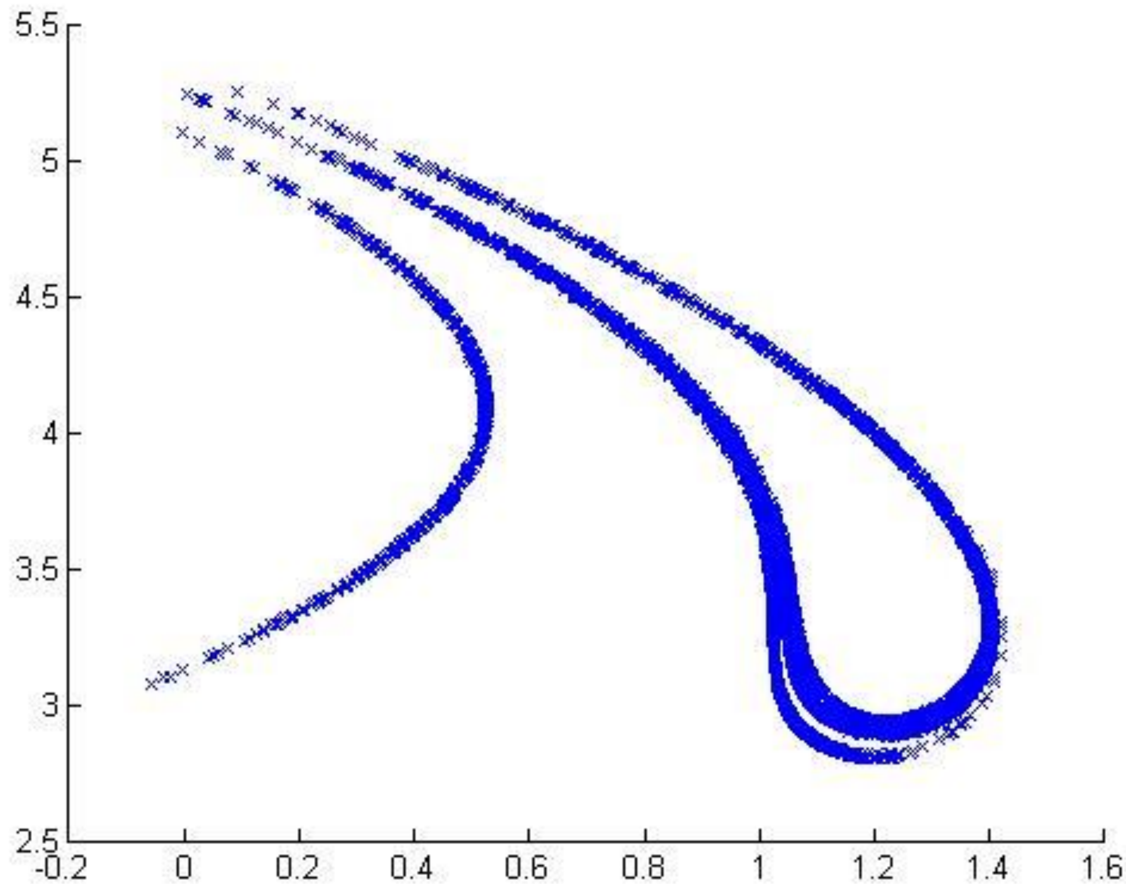

Fig8:Poincare section for a=3.5

We have calculated Poincare section for Mathieu equation. One(fig6) is calculated for a=2.7.As expected before we get only very few points in the Poincare section. That means it is a limit cycle regime. We have seen earlier for a=2.7 we get a limit cycle in phase space. Another (Fig8) is calculated for a=3.5.Here we get densely filled points. It is a strong indication of the presence of chaos.

Bifurcation Theory

Bifurcation is a critical theory to analyse systems qualitative and quantitative behavior with respect to any parameter value. Bifurcation means the qualitative change of any system with respect to any control parameter. Where we get a linear instability we get bifurcation. So if there is a change in the trajectory structure of the solution with respect to the parameter value bifurcation occurs.

We have calculated bifurcation first of a simple logistic system then for single Mathieu oscillator. It is a very time taking process to plot bifurcation diagram as there are humongous number of points we have plotted against each parameter value

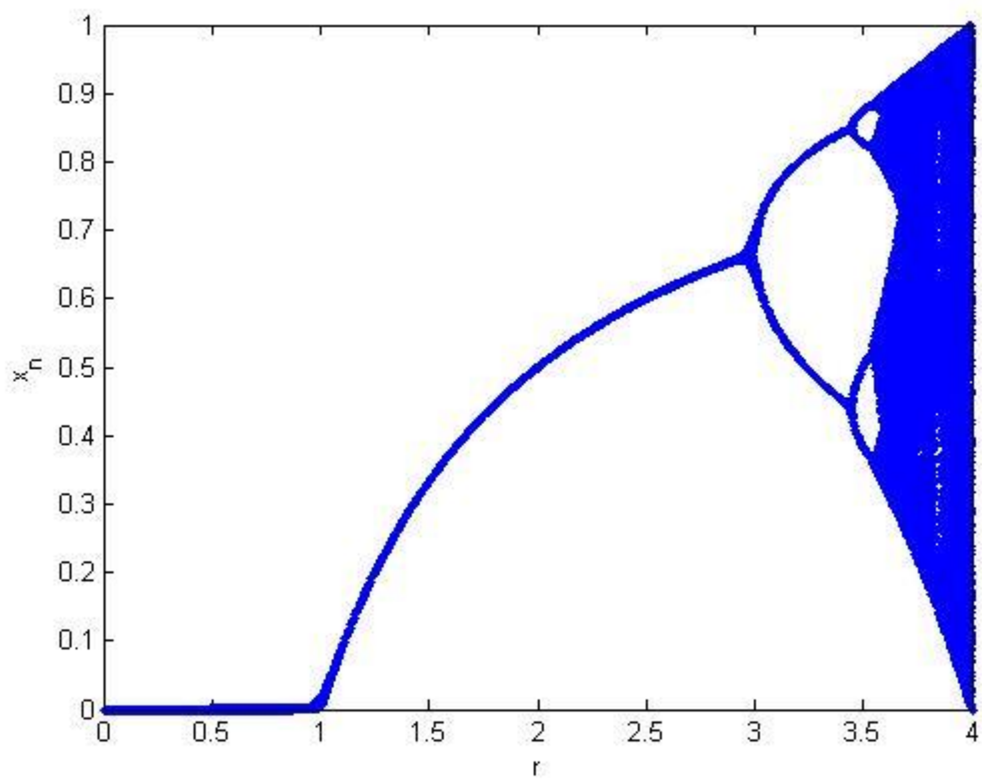

Fig9:Bifurcation for the logistic map.

Here we have plotted points which are lying on the stable orbits. So for r=3.2 we have two such points. Here what happens is called period doubling. In period doubling what happens is suddenly the period of the system becomes double of its previous value as the system parameter is slightly changed. It means a new limit cycle is born with a double period. Now if it continues to happen sequentially then we can say the system slowly becomes chaotic. So period doubling is a path to chaos .Now for r=3.5 there are 4 such points .For r=3.6 there 8 such points and it continues like this. For r greater than 3.9 we see many points are there and from there the system is entering into the chaotic region.

We have also plotted the bifurcation diagram for Mathieu oscillator. Here also period doubling happens. For a=3.2 and then slowly it enters into the region of chaos. This kind of bifurcation is called period doubling bifurcation.

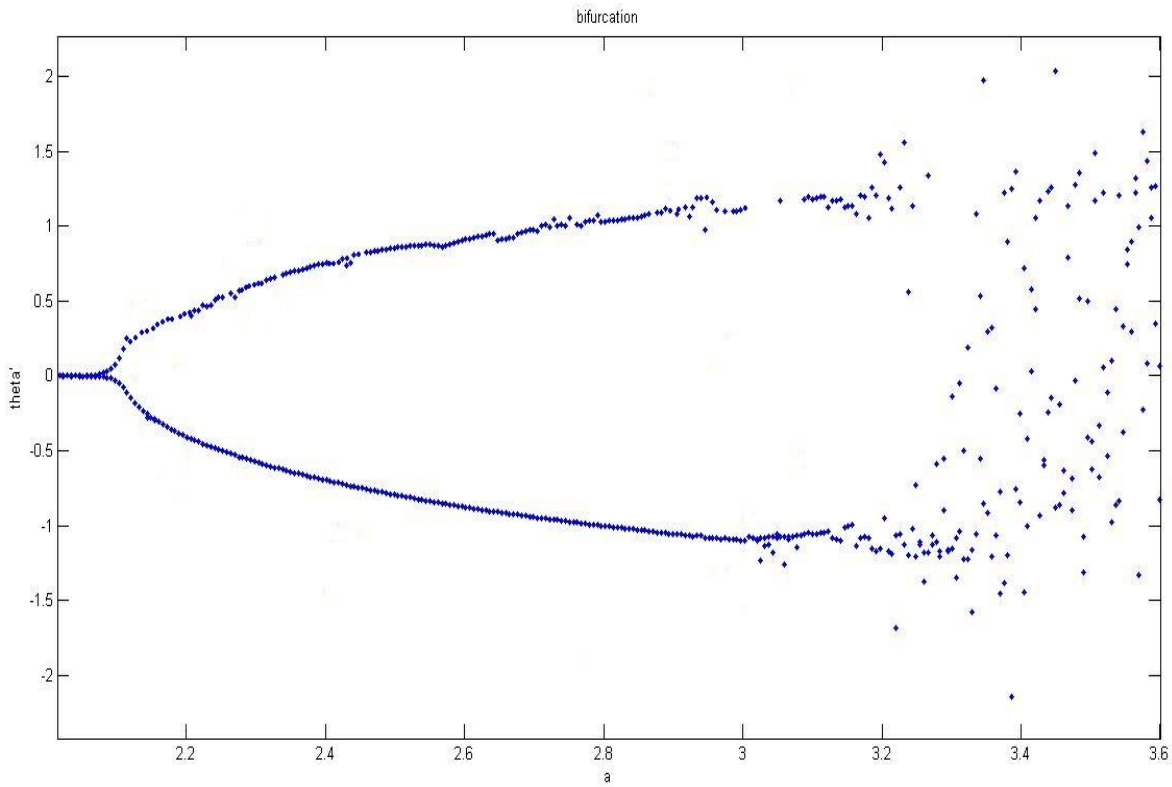

Fig10:bifurcation for Mathieu oscillator

RESULTS

We have got excellent results in this paper. All methods to detect chaos was successfully tested. We have seen that Mathieu acts as a chaotic for its parameter value ‘a’ greater than 3.2. For logistic equation we got chaos for ’r’ greater than 3.6.We also have found an essential relation between bifurcation and Lyapunov Exponents. For what value of parameter we get a peak in the Lyapunov Exponents we get a period doubling in the bifurcation diagram. For logistic equation at r=3.2 we get period doubling in the bifurcation diagram and a peak in the LE diagram. At r=3.5 we get four such points. So it is a significant discovery.

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MATLAB CODES

TO SOLVE MATHIEU EQUATION

```
% The object of this code is to find the
graph of a chaotic SINGLE Mathieu
% Oscillator
function SOL=single_mathieu(t,X)
    x=X(1);y=X(2);z=X(3);
    lambda= 0.5;
    omega=2;
    a=3.5;
    SOL(1)=y;
    SOL(2)=-2*lambda*y-(1+a*cos(z))*sin(x);
    SOL(3)=omega;

    SOL=SOL';
end
```

POINCARE SECTION

```
% This finds the Poincare Map for 3D phase
space. The section is always a
% Plane
% everything vector should row
% count is the number of intersections
% pts is the vector of points
% xout, yout and zout are the points in the
phase space.
% S is the function handler for the plane.
% p0 is a point on the plane and n is the
vector normal to plane like [1 0
```

```
% 0] is i_cap if the plane is parallel to
yz plane.
% plt is a vector. Suppose the section is
x=2, plt=[2 3] i.e. the plane
% parallel to it
function
[count,pts]=poincareSec(xout,yout,zout,S,n,
p0,plt)
    close all;
    hold on;
    count=0;
% Round the numbers to 4 digits
    xout=roundArray(xout,4);
    yout=roundArray(yout,4);
    zout=roundArray(zout,4);
    pts=[];
for j=1:length(xout)-1
if(S([xout(j) yout(j)
zout(j)])<0&&S([xout(j+1) yout(j+1)
zout(j+1)])>0)
% Have a line drawn between the two points.
This is not the
% worst of interpolations if you have many
points.
            x1=[xout(j) yout(j)
zout(j)];x2=[xout(j+1) yout(j+1)
zout(j+1)];
            l=x2-x1;
            l0=x2;

            d=dot((p0-l0),n)/dot(l,n);
            p=d*l+l0;
```

```
            plot(p(plt(1)),p(plt(2)),'r');
            pts=[pts;p(plt(1)),p(plt(2))];
            count=count+1; hold on;
%pause(.5); % Keep this if you like to see
a movie
end
end

   hold off;
end

function ret=roundArray(x,digs)
        ret=round(x*(10^digs))/(10^digs);

end
```

LYAPUNOV EXPONENTS

```
for i = 3000:4000
    r(i) = 0.001*i;
for j = 1:499
            x(j+1) = r(i)* x(j)* (1-x(j));
end
        y(1) = x(500);
for j = 1:9999
            y(j+1) = r(i)* y(j)* (1-y(j));
            l(j+1) = log(abs(r(i)-
2*r(i)*y(j+1)));
end
        lav(i) = sum(l)/10000;
end

plot(r,lav,'-')
```

```
xlim([3 4])
xlabel('r')
ylabel('le')
```

BIFURCATION CODE

LOGISTICS

```
x(1) = 0.5;

for i = 0:800
    r = 0.005*i;
    a(200*i+1:200*(i+1)) = r;
for j = 1:249
            x(j+1) = r* x(j)* (1-x(j));
end
for k = 1:200

            b(200*i+k) = x(k+50);
end
end

plot(a,b,'.')
xlabel('r')
ylabel('x_n')
```

MATHIEU EQUATION

```
clc;close all; clear;
global a;

a_arr=linspace(2,4,250);
L_arr=zeros(0,length(a_arr));
```

```
step_s=599; %change this to see if it turns
out well.
for i=1:length(a_arr)
    a=a_arr(i);

    [~,L]=ode45(@single_mathieu_bifurc,[0
1500],[0,0.2,2]);
%fprintf('%d\n',i);

    size_temp=size(L(end-
600:step_s:end,1));
    a_arr2(1:size_temp)=a_arr(i);
    L_arr(1:size_temp)=L(end-
600:step_s:end,1);

    plot (a_arr2,L_arr,'.');
    hold on;
%L_arr(1,400*(i-1)+1:400)=L(end-399:end,1);
%this will plot bifurcation at x

%clear t L;
end
%plot (a_arr2,L_arr);
```