Numerical Solution of Interval Nonlinear System of Equations

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IN

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By

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DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "Numerical Solution of Interval Nonlinear System of Equations" in partial fulfilment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology Rourkela is an authentic record of my own work carried out under the supervision of Prof. S. Chakraverty.

The matter embodied in this thesis has not been submitted in other institution or university for the award of any other degree or diploma.

(SANDEEP NAYAK)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ABSTRACT

Nonlinear system of equations has a wide range of applications in both science and engineering. But usually the variables and parameters may not be found as crisp numbers. So, the nonlinear system of equations have been considered in interval form. The basic concepts about interval arithmetic has been discussed. Three methods has been presented to solve interval nonlinear system of equations followed by some examples. The interval form is converted to crisp system of equations by the help of vertex method and then the same may be solved by Newton's method. The Krawczyk method has been presented to solve these type of problems too. Finally, a new method has been proposed which is computationally efficient than vertex method. Accordingly some examples have been solved using all the methods to show the comparison and reliability of the methods.

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Chapter 1

INTRODUCTION

The system of nonlinear equations has a wide range of applications in various fields of science and engineering such as operational research, physics, economics, civil and electrical engineering, and social sciences. Solution of a nonlinear system of equations has been investigated using various methods when the involved variables of the system are in crisp numbers. The variables or data are taken as crisp numbers for the sake of easy calculation and simplicity. However, getting the variables as crisp numbers are not possible in real life problems. Most of the variables are found by some experiments with some experimental uncertainty associated with the experimental procedures. For example, measuring length of a wire by ruler, we cannot get a crisp number. In particular, the measured value depends on the viewing angle of the ruler and on the person measuring it. Therefore these variables should be taken as intervals while solving the governing equations.

Interval computation has been developed by mathematicians since nineteen fifties. It gives an approach to putting bounds on errors in mathematical computation and thus developing numerical methods that yield reliable results. The first application of interval computation was presented by R. E. Moore in 1959. The main ideas of interval computation appeared at Stanford in 1962. Later, the centre of interval computations moved to Europe, mainly to Germany. The first specialized journal for interval computation was from Germany. Interval arithmetic can be used in various areas such as set inversion, motion planning, set estimation or stability analysis. It is also used in rounding error analysis, tolerance analysis, and fuzzy interval arithmetic. Interval computation is applied to study uncertainty analysis. It is also applied in the field of mathematical modelling and industrial mathematics, numerical analysis, data processing, expert systems, control and optimization.

Interval nonlinear systems of equations have been studied by few authors which are mentioned in this paper. The method of “interval extended zero” for solving interval equations has been studied by Sevastjanov and Dymova [1]. The modification of Newton’s method with the property that the sequence of interval widths is dominated by a quadratically convergent sequence has been discussed by Madsen [2]. An interval version of Newton’s method too has been discussed by Moore [3]. A computational test for convergence of iterative methods for nonlinear systems has been studied by Moore [4]. The vertex method and another proposed method for linear system of equations has been discussed by Chakraverty and Nayak [5]. A new approach to solve fuzzy linear system of equations where the coefficient matrix is crisp and the solution and right hand side vectors are intervals has been proposed by Chakraverty and Behara [6]. Solution to fully fuzzy system of linear equations using single and double parametric form of fuzzy numbers has also been proposed by Chakraverty and Behera [7].

Efforts have been made by various researchers to solve interval nonlinear system of equations but a lot of important information has not been considered in the existing literature. Moreover, some of the methods are not computationally efficient. The purpose of the project is to fill these gaps. The aim of the project is to develop some methods for solving interval nonlinear system of equations. Some methods have been developed to solve these type of problems. All the methods have been validated by considering different types of examples. In one application problem the solution has been compared with the known results that are found in literature.

The rest of the thesis is organized as follows: In Chapter 2 we have discussed basic concepts of interval arithmetic and Newton's method. Some methods has been developed to solve interval nonlinear system of equation in Chapter 3. In Chapter 4 we solve different types of problems that validate our developed methods. In Chapter 5 we conclude our project work with some results. Finally some suggestions for future work are mentioned followed by references.

Chapter 2

PRELIMINARIES

This chapter presents the definition of interval, basic concepts of interval arithmetic and some properties of interval arithmetic. Some properties of interval matrix are also discussed in this chapter. The basic concepts and properties of intervals may be found in [8] and [9].

2.1 Interval

Let $a, b \in R$ such that $a \leq b$. A interval $[a, b]$ is defined as

$$[a, b] = \{y \in R \mid a \leq y \leq b\}$$

Where, a is the lower bound and b is the upper bound of $[a, b]$.

2.2 Equality of Intervals

The two interval numbers $A = [a, b]$ and $B = [c, d]$ are said to be equal if their end points i.e. bounds are equal.

$$[a, b] = [c, d] \Leftrightarrow a = b \text{ and } c = d$$

2.3 Interval Arithmetic

Let $A = [a, b]$ and $B = [c, d]$ be any two intervals. Then

- $A + B = [a + c\,,\ b + d]$
- $A - B = [a - d\,,\ b - c]$
- $A.B = [\min\{ac, ad, bc, bd\},\ \max\{ac, ad, bc, bd\}]$
- $A/B = [a, b].[1/c\,, 1/d]$

2.4 Reciprocal of Interval

For any zeroless interval $[a, b]$, the reciprocal of interval is given by:

$$[a, b]^{-1} = [b^{-1}, a^{-1}]$$

2.5 Negation of Interval

For any interval $[a, b]$, interval negation is given by

$$-[a, b] = [-b, -a]$$

2.6 Interval Width

The width of any interval $[a, b]$ is given by

$$w([a, b]) = b - a$$

2.7 Interval Radius

The radius of any interval $[a, b]$ is given by

$$r([a, b]) = \frac{w([a, b])}{2} = \frac{b - a}{2}$$

2.8 Interval Midpoint

The midpoint of any interval $[a, b]$ is given by

$$m([a, b]) = \frac{b + a}{2}$$

2.9 Commutativity of Intervals

Both interval addition and multiplication are commutative.

$$X + Y = Y + X \qquad \forall\, X, Y \in [R]$$

$$X \times Y = Y \times X \qquad \forall\, X, Y \in [R]$$

2.10 Associativity of Intervals

Both interval addition and multiplication are associative.

$$(X + (Y + Z)) = ((X + Y) + Z) \qquad \forall\, X, Y, Z \in [R]$$

$$(X \times (Y \times Z)) = ((X \times Y) \times Z) \qquad \forall\, X, Y, Z \in [R]$$

2.11 Cancellativity Laws in Interval Arithmetic

Interval addition is cancellable.

$$X + Z = Y + Z \ \Rightarrow X = Y \qquad \forall\, X, Y, Z \ \in [R]$$

An interval is cancellable for multiplication if, and only if, it is a zeroless interval.

$$(X \times Z = Y \times Z \ \Rightarrow X = Y) \ \Leftrightarrow 0 \notin Z \qquad \forall\, X, Y, Z \ \in [R]$$

2.12 Subdistribituvity of Intervals

The distributive law does not always hold in classical interval arithmetic. The distributive law

$$Z \times (X + Y) = Z \times X + Z \times Y$$

holds if, and only if,

i. Z is a point interval
ii. $X = Y = [0,0]$
iii. $xy \geq 0 \ \forall\, x \in X, \forall\, y \in Y$

In general, the subdistributive law holds for in classical interval arithmetic

$$Z \times (X + Y) \subseteq Z \times X + Z \times Y \qquad \forall\, X, Y, Z \ \in [R]$$

Some properties of interval matrix have been discussed in [9].

For any interval matrices $A, B \; and \; C$, the matrices doesn't follow the commutativity and associativity of multiplication.

$$A \times B \neq B \times A$$

$$((A \times B) \times C) \neq (A \times (B \times C))$$

2.13 Inclusion Isotonicity

The function $F = F(X_1, X_2, \dots, X_n)$ is inclusion isotonic if

$$Y_i \subseteq X_i \text{ for } i = 1,2, \dots, n$$

$$\Rightarrow \ F(Y_1, Y_2, \dots, Y_n) \subseteq F(X_1, X_2, \dots, X_n)$$

2.14 Newton's Method

Newton's method can be applied to solve crisp nonlinear system of equations. The method has the same concept as Newton Raphson method but in matrix form. The method may be found in any standard book such as [10] and [11].

Let us consider the finite system of nonlinear equations

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0 \\ f_2(x_1, x_2, \dots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0 \end{aligned} \tag{1}$$

Now put the above equations in the following matrix form

$$\{F(x)\} = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2}$$

Here $\{F(x)\}$ is a column vector which is given as

$$\{F(x)\} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

with $(x) = (x_1, x_2, \dots, x_n)$

Now, the concept of Newton Raphson method to get the roots of a single nonlinear equation is as follows

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots = 0 \tag{3}$$

Neglecting higher order terms in (3) and equating to zero

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{4}$$

which gives the iterative formula of Newton Raphson method as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{5}$$

The same concept can be extended to system of nonlinear equations. Let the initial vector be

$$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$$

Expanding the function $f_1(x)$ about the above initial guess, we have

$$f_1(x) = f_1(x^{(0)}) + \left[\frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \dots, \frac{\partial f_1}{\partial x_n}\right]\{x - x^{(0)}\} + \text{higher order terms} \qquad (6)$$

Neglecting higher order terms, we can write $\{F(x)\}$ as

$$\{F(x)\} = \{F(x^{(0)})\} + [J]\{x - x^{(0)}\} = 0 \qquad (7)$$

where $[J]$ is the Jacobian matrix is given by

$$[J] = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ & \vdots & & \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

So the iterative procedure of Newton's method is given by

$$[x_{k+1}] = [x_k] - [J(x_k)]^{-1}F(x_k) \qquad (8)$$

where

$$[J(x_k)]^{-1} = \frac{1}{detJ(x_k)}[cofactor\ matrix\ of\ J(\,x_k)]^T$$

The convergence check is performed using the norms of the vectors x_{k+1} and x_k to find the relative error.

$$\xi = \frac{\|x_{k+1} - x_k\|_p}{\|x_{k+1}\|_p}$$

The p value for the norm is decided depending on the nature of the problem. If the value of ξ is less than or equal to some predefined tolerance, then x_{k+1} is the solution of the nonlinear system of equations.

Chapter 3

DEVELOPED METHODS

This chapter presents the problem that has been considered in the project work. Two methods have been developed to solve these type of problems. A new method has also been proposed to solve these interval nonlinear system of equations.

3.1 Interval Nonlinear System of Equations

Let us consider finite system of interval nonlinear equations

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= [\underline{b_1}, \overline{b_1}] \\ f_2(x_1, x_2, \dots, x_n) &= [\underline{b_2}, \overline{b_2}] \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= [\underline{b_n}, \overline{b_n}] \end{aligned} \tag{9}$$

Here the coefficient matrix is crisp and the solution will be obtained as intervals.

The following sections of this chapter presents some methods to solve the above problem.

For a finite interval nonlinear system of equations first the combinations of lower bound with lower bound and upper bound with upper bound are taken. Solving those two combinations and taking minimum and maximum of the two solutions as lower and upper bound, we may have a solution. For example let us consider two nonlinear system of equations with two unknowns as

$$\begin{aligned} x_1^2 + x_2 &= [10.9, 11.1] \\ x_1 + x_2^2 &= [6.9, 7.1] \end{aligned} \tag{10}$$

By applying the above procedure to equation (10) we may have,

$$\begin{aligned} x_1^2 + x_2 &= 10.9 \\ x_1 + x_2^2 &= 6.9 \end{aligned} \tag{11}$$

and

$$\begin{aligned} x_1^2 + x_2 &= 11.1 \\ x_1 + x_2^2 &= 7.1 \end{aligned} \tag{12}$$

Solving the above two systems of nonlinear equations (11) & (12), we will get

$$x_1 = [2.9869, 3.0130]$$

$$x_2 = [1.9781, 2.9869]$$

But one may very well see this type of procedure will not work which may be understood by taking a counter example. Suppose, we take real numbers 11.08 and 6.91 in the right hand sides of equation. It may be noted that $11.08 \in [10.9, 11.1]$ and $6.91 \in [6.9, 7.1]$. Accordingly we will now solve the following nonlinear (crisp) system of equations

$$\begin{aligned} x_1^2 + x_2 &= 11.08 \\ x_1 + x_2^2 &= 6.91 \end{aligned} \tag{13}$$

The solution of the nonlinear system of equations (13) is

$$x_1 = 3.0178$$

$$x_2 = 1.9729$$

It may be seen that the solution does not lie in the obtained interval. So, the combinations taken in the above procedure fails to satisfy the solution.

3.2 Vertex Method

In this context, Dong and Shah [8] introduced an excellent method known as vertex method for interval system of linear equations. In this method all the combinations of lower and upper bounds of the intervals are taken into consideration. For N interval parameters this corresponds to 2^N combinations. Out of all the computations the minimum and maximum are selected as solution for each unknown. In this project the same method is extended for interval system of nonlinear equations with the help of Newton's method.

For clear understanding we consider two nonlinear systems of equations with two unknowns. So, we have

$$\begin{aligned} f_1(x_1, x_2) &= [\underline{b_1}, \overline{b_1}] \\ f_2(x_1, x_2) &= [\underline{b_2}, \overline{b_2}] \end{aligned} \tag{14}$$

Now considering all the possible combinations of lower bounds and upper bounds, we have

$$\begin{aligned} f_1(x_1, x_2) &= \underline{b_1} \\ f_2(x_1, x_2) &= \underline{b_2} \end{aligned} \tag{15}$$

$$\begin{aligned} f_1(x_1, x_2) &= \underline{b_1} \\ f_2(x_1, x_2) &= \overline{b_2} \end{aligned} \tag{16}$$

$$\begin{aligned} f_1(x_1, x_2) &= \overline{b_1} \\ f_2(x_1, x_2) &= \underline{b_2} \end{aligned} \tag{17}$$

$$\begin{aligned} f_1(x_1, x_2) &= \overline{b_1} \\ f_2(x_1, x_2) &= \overline{b_2} \end{aligned} \tag{18}$$

Solving above four systems of nonlinear equations from (15) to (18) by Newton's method, we will get four values of x_1 and x_2. Out of those four values the minimum and maximum for each unknown variable are chosen and those values will be the lower and upper bounds of the interval solution respectively. The same concept can be extended for more number of variables.

3.3 Krawczyk Method

The theorems and the computational test for the convergence of Krawczyk method may be found in [3], [8] and [13].

Theorem 1 ([3], [8] and [13])
Let Y be a non-singular real matrix approximating the inverse of the real Jacobian matrix $F'(m(X))$ with elements $F'(m(X))_{ij} = \frac{\partial f_i(x)}{\partial x_j}$ at $x = m(X)$. Let y be a real vector contained in the interval vector X. Define $K(X)$ by

$$K(X) = y - Yf(y) + \{I - YF'(X)\}(X - y) \tag{19}$$

If $K(X)$ lies in X, then the nonlinear system of equations has a solution in $K(X)$.

If the interval vector $X = (X_1, \cdots, X_n)$ is an n-cube so that $w(X_i) = w(X)$ for $i = 1,2, \dots, n$, and if we choose $y = m(X)$, then $K(X)$ lies in the interior of X if

$$\|K(X) - m(X)\| < w(X)/2 \tag{20}$$

Theorem 2 ([3], [8] and [13])
Let $X = (X_1, \cdots, X_n)$ be an n-cube, $y = m(X)$, and Y a non-singular real matrix. Suppose the above condition (20) is satisfied. Put $X^{(0)} = X, Y^{(0)} = Y$ and consider an arbitrary real vector $x^{(0)}$ in $X^{(0)}$. Then the system has a unique solution in X, and the following iteration method converges to the solution:

$$X^{(k+1)} = X^{(k)} \cap K(X^{(k)}) \qquad (k = 1,2, \dots, n)$$

where

$$K(X^{(k)}) = y^{(k)} - Y^{(k)} f\big(y^{(k)}\big) + \{I - Y^{(k)} F'(X^{(k)})\} Z^{(k)} \tag{21}$$

and

$$y^{(k)} = m\big(X^{(k)}\big), \quad Z^{(k)} = X^{(k)} - m(y^{(k)})$$

where $Y^{(k)}$ is chosen as

$$Y^{(k)} = \begin{cases} Y & if \ \|I - YF'(X^{(k)})\| \leq \|I - Y^{(k-1)}F'(X^{(k-1)})\| \\ Y^{(k-1)} & otherwise \end{cases}$$

here Y is an approximation to $[m(F'(X^{(k)}))]^{-1}$.

In the case of interval nonlinear system of equations inclusion isotonic functions F and F' are required such that $f(x)$ is contained in $F(x)$ and $f'(x)$ in $F'(x)$ for every choice of real constants in the interval coefficients and for every x in X. In the second theorem $f(y^{(k)})$ is replaced by $F(y^{(k)})$. The iterative equation (21) becomes

$$K(X^{(k)}) = y^{(k)} - Y^{(k)}F\left(y^{(k)}\right) + \{I - Y^{(k)}F'(X^{(k)})\}Z^{(k)}. \quad (22)$$

3.4 Proposed Method

The interval is converted to parametric form by using some parameter β. Let $A = [a, b]$ be any interval, then the parametric form of A can be written as

$$A = \beta(b - a) + a \tag{23}$$

where $0 \leq \beta \leq 1$

Substituting $\beta = 0$ and 1, the lower and upper bounds of the interval can be obtained respectively.

The intervals are converted to parametric form in the interval nonlinear system of equations using suitable parameters for each interval. The solutions are obtained in the parametric form. Then by substituting the value of parameters, all the required solutions can be obtained. Finally we take the maximum and minimum as lower and upper bounds of the solution.

Again let us consider the system of equations (14)

$$f_1(x_1, x_2) = [\underline{b_1}, \overline{b_1}]$$

$$f_2(x_1, x_2) = [\underline{b_2}, \overline{b_2}]$$

Now we convert it into parametric form

$$\begin{aligned} f_1(x_1, x_2) &= \beta_1 \left(\overline{b_1} - \underline{b_1}\right) + \underline{b_1} \\ f_2(x_1, x_2) &= \beta_2 \left(\overline{b_2} - \underline{b_2}\right) + \underline{b_2} \end{aligned} \tag{24}$$

Where $0 \leq \beta_1, \beta_2 \leq 1$

Solving the above two nonlinear equations (24) we can get x_1 and x_2 in terms of β_1 and β_2. Then by substituting the values of parameters β_1 and β_2, we can get the required solutions. The maximum and minimum are taken as lower and upper bound of the solution. This method can be extended to more variables by taking equal number of parameters as unknown variables.

Chapter 4

NUMERICAL EXAMPLES

4.1 Examples Solved by Vertex method

4.1.1 Example 1

Let us consider the equations

$$\begin{aligned} x_1^2 + x_2 &= [10.9,11.1] \\ x_1 + x_2^2 &= [6.9,7.1] \end{aligned} \tag{25}$$

Considering all the four possible combinations of lower and upper bounds and solving them through the MATLAB code with initial approximation $x^{(0)} = (1,1)^T$, we have

$$\underline{x_1} = \min\{2.9869,2.9782,3.0130,3.0217\} = 2.9782$$

$$\overline{x_1} = \mathrm{m}ax\{2.9869,2.9782,3.0130,3.0217\} = 3.0217$$

$$\underline{x_2} = \min\{1.9781,2.0302,2.0216,1.9693\} = 1.9693$$

$$\overline{x_2} = \mathrm{m}ax\{1.9781,2.0302,2.0216,1.9693\} = 2.0302$$

So the solution of (25) in interval form is

$$x_1 = [2.9782,3.0217]$$

$$x_2 = [1.9693,2.0302]$$

4.1.2 Example 2

Accordingly we consider the equations as

$$\begin{aligned} x_1^2 + x_2 &= [36.85,37.05] \\ x_1 + x_2^2 &= [6.95,7.1] \\ x_1 + x_2 + x_3 &= [10.9,11.15] \end{aligned} \tag{26}$$

Considering all the eight possible combinations of lower and upper bounds and solving them through the MATLAB code with initial approximation $x^{(0)} = (1,1,1)^T$, , we have

$$\underline{x_1} = \min\{5.9891, 5.9827, 6.0066, 6.0001\} = 5.9827$$

$$\overline{x_1} = \mathrm{m}ax\{5.9891, 5.9827, 6.0066, 6.0001\} = 6.0066$$

$$\underline{x_2} = \min\{0.9802, 1.0570, 0.9713, 1.0488\} = 0.9713$$

$$\overline{x_2} = \mathrm{m}ax\{0.9802, 1.0570, 0.9713, 1.0488\} = 1.0570$$

$$\underline{x_3} = \min\begin{Bmatrix}3.9306, 3.8603, 4.1806, 4.1103, \\ 3.9221, 3.8511, 4.1721, 4.1011\end{Bmatrix} = 3.8511$$

$$\overline{x_3} = \mathrm{m}ax\begin{Bmatrix}3.9306, 3.8603, 4.1806, 4.1103, \\ 3.9221, 3.8511, 4.1721, 4.1011\end{Bmatrix} = 4.1806$$

Accordingly the solution of (26) is found to be

$$x_1 = [5.9827, 6.0066]$$

$$x_2 = [0.9713, 1.0570]$$

$$x_2 = [3.8511, 4.1806]$$

4.1.3 Example 3

Let us consider now the equations as

$$\begin{aligned} x_1^2 + x_2 + x_3^2 &= [13.8, 14.2] \\ x_1^2 + x_2^2 + x_3 &= [7.85, 8.15] \\ x_1 + x_2^2 + x_3^2 &= [11.9, 12.1] \end{aligned} \tag{27}$$

Considering all the eight possible combinations of lower and upper bounds and solving them through the MATLAB code with initial approximation $x^{(0)} = (1,1,1)^T$, , we have

$$\underline{x_1} = \min\begin{Bmatrix}1.9657, 1.9232, 1.9604, 2.0009, \\ 2.0449, 2.0012, 2.0366, 2.0744\end{Bmatrix} = 1.9232$$

$$\overline{x_1} = \mathrm{m}ax\begin{Bmatrix}1.9657, 1.9232, 1.9604, 2.0009, \\ 2.0449, 2.0012, 2.0366, 2.0744\end{Bmatrix} = 2.0744$$

$$\underline{x_2} = \min\begin{Bmatrix}0.9982, 1.0706, 1.1578, 1.0938, \\ 0.7945, 0.8917, 1.0020, 0.9226\end{Bmatrix} = 0.7945$$

$$\overline{x_2} = \mathrm{m}ax\begin{Bmatrix}0.9982, 1.0706, 1.1578, 1.0938, \\ 0.7945, 0.8917, 1.0020, 0.9226\end{Bmatrix} = 1.1578$$

$$\underline{x_3} = \min\begin{Bmatrix}2.9896, 3.0051, 2.9663, 2.9500, \\ 3.0371, 3.0502, 3.0104, 2.9957\end{Bmatrix} = 2.9500$$

$$\overline{x_3} = \mathrm{m}ax\begin{Bmatrix}2.9896, 3.0051, 2.9663, 2.9500, \\ 3.0371, 3.0502, 3.0104, 2.9957\end{Bmatrix} = 3.0502$$

Then the final solution of (27) in interval form are

$$x_1 = [1.9232, 2.0744]$$

$$x_2 = [0.7945, 1.1578]$$

$$x_2 = [2.9500, 3.0502]$$

4.1.4 An Application Problem

Example 4 (Electrical Circuit Analysis)

An electrical circuit analysis problem in 3 variables has been considered here [14]. This is a well-known network used to realize a third order butterworth function [15]. The value of R is chosen as 0.5 ohms and $x_1 = C_1, x_2 = C_2, x_3 = C_3$. But in this paper the nonlinear system of equations are considered in terms of intervals. So, the design equations in intervals are considered as

$$\begin{gathered} x_1 + 2x_2 + x_3 = [5.7,6.3] \\ 2x_1x_2 + x_2x_3 = [5.6,6.4] \\ x_1x_2x_3 = [2.5,3.5] \end{gathered} \tag{28}$$

Considering again all the eight possible combinations of lower and upper bounds and solving them (using vertex method) through the MATLAB code with initial approximation $x^{(0)} = (1,1,1)^T$, we have

$$\underline{x_1} = \min\begin{Bmatrix}1.0144,1.9122,1.5199,2.0101, \\ 0.6269,1.0487,0.9041,1.0107\end{Bmatrix} = 0.6269$$

$$\overline{x_1} = \mathrm{m}ax\begin{Bmatrix}1.0144,1.9122,1.5199,2.0101, \\ 0.6269,1.0487,0.9041,1.0107\end{Bmatrix} = 2.0101$$

$$\underline{x_2} = \min\begin{Bmatrix}1.5455,0.98761.5643,1.1586, \\ 1.2858,1.0787,2.0102,1.4530\end{Bmatrix} = 0.9876$$

$$\overline{x_2} = \mathrm{m}ax\begin{Bmatrix}1.5455,0.98761.5643,1.1586, \\ 1.2858,1.0787,2.0102,1.4530\end{Bmatrix} = 2.0102$$

$$\underline{x_3} = \min\begin{Bmatrix}1.5946,1.8446,1.0515,1.4862, \\ 3.1015,3.0938,1.3755,2.3833\end{Bmatrix} = 1.0515$$

$$\overline{x_3} = \mathrm{m}ax\begin{Bmatrix}1.5946,1.8446,1.0515,1.4862, \\ 3.1015,3.0938,1.3755,2.3833\end{Bmatrix} = 3.1015$$

The solution of (28) in interval form is found to be

$$x_1 = [0.6269,2.0101]$$

$$x_2 = [0.9876,2.0102]$$

$$x_3 = [1.0515,3.1015]$$

Now with another initial approximation $x^{(0)} = (4,0,1)^T$, the solution of (28) is found to be

$$x_1 = [1.9122, 4.0703]$$

$$x_2 = [0.6121, 1.1586]$$

$$x_3 = [0.8661, 1.8446]$$

It may be seen that both the operating points of the butterworth function problem [14] lie in the obtained intervals using vertex method.

4.2 Example solved by Krawczyk Method

Example 5

Let us consider the equations

$$\begin{aligned} x_1^2 + x_2 &= [10.9,11.1] \\ x_1 + x_2^2 &= [6.9,7.1] \end{aligned} \tag{29}$$

The interval form of F and F' can be written as

$$F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \end{pmatrix}$$

Where

$$\begin{aligned} F_1(X) &= X_1^2 + X_2 - [10.9,11.1] \\ F_2(X) &= X_1 + X_2^2 - [6.9,7.1] \end{aligned} \tag{30}$$

and

$$F'(X) = \begin{pmatrix} 2X_1 & 1 \\ 1 & 2X_2 \end{pmatrix} \tag{31}$$

Take initial guess as

$$X = \begin{pmatrix} [2.5,3.5] \\ [1.5,2.5] \end{pmatrix}$$

From equation (31) we have

$$m(F'(X)) = \begin{pmatrix} m([5,7]) & m([1,1]) \\ m([1,1]) & m([3,5]) \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix}$$

The approximate inverse of the matrix $m(F'(X))$ is

$$Y = \begin{pmatrix} 0.17 & -0.04 \\ -0.04 & 0.26 \end{pmatrix}$$

So, we get

$$y - YF(y) = \begin{pmatrix} [2.79,3.21] \\ [1.7,2.3] \end{pmatrix}$$

$$(I - YF'(X))(x - y) = \begin{pmatrix} [-0.12,0.12] \\ [-0.16,0.16] \end{pmatrix}$$

From the iterative procedure (22) we will get

$$K(X) = \begin{pmatrix} [2.67,3.33] \\ [1.54,2.46] \end{pmatrix}$$

So the first iteration is found to be

$$X^{(1)} = \begin{pmatrix} [2.67,3.33] \\ [1.54,2.46] \end{pmatrix}$$

Next four iterations are given in the table

Table 1 Iterative values of X_1 and X_2

No. of iterations	X_1	X_2
0	[2.5,3.5]	[1.5,2.5]
1	[2.67,3.33]	[1.54,2.46]
2	[2.7248,3.2752]	[1.5746,2.4254]
3	[2.74,3.26]	[1.5943,2.4057]
4	[2.744592,3.255408]	[1.603804,2.396196]

The iterative values of each unknown variable X_1 and X_2 are given in Table 1 for first four iterations along with the initial guess. From the values in the table it can be clearly observed that the initial guess converges to the solution.

The convergence of X_1 and X_2 are shown in Fig. 1 and Fig. 2

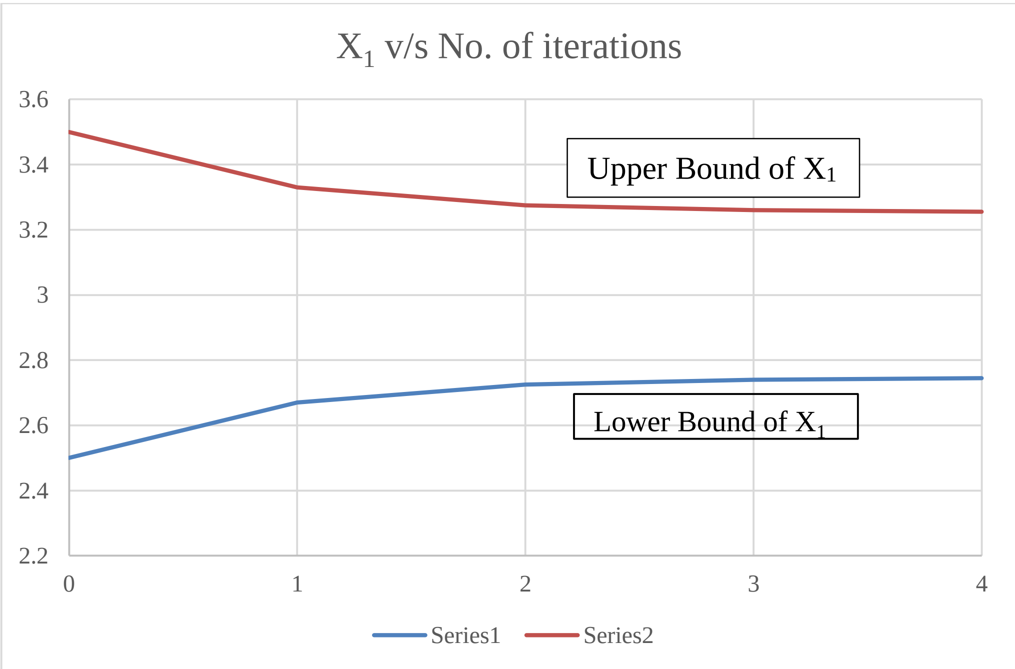

Fig. 1 Convergence of X_1

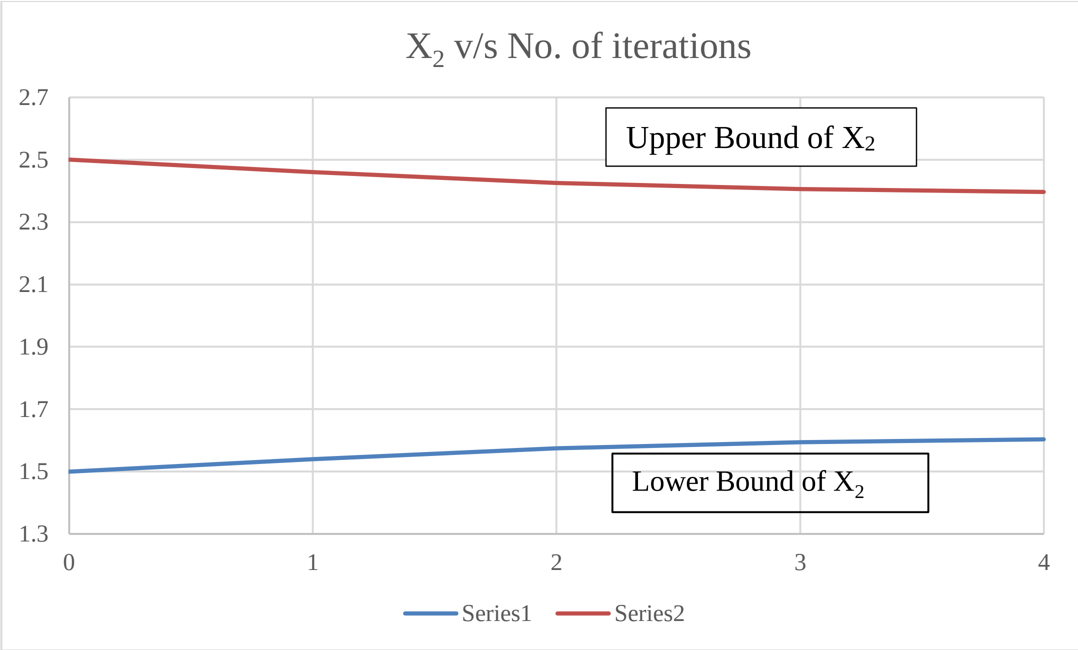

Fig. 2 Convergence of X_2

The Figs. 1 & 2 represent the plot between number of iterations and the solution of X_1 and X_2 respectively. In both the cases the convergence of lower and upper bounds of solution have been shown for X_1 and X_2.

4.3 Examples Solved by Proposed Method

4.3.1 Example 6

Let us consider the equations

$$\begin{gathered} x_1^2 + x_2 = [10.9,11.1] \\ x_1 + x_2^2 = [6.9,7.1] \end{gathered} \tag{32}$$

Now converting the intervals into crisp, we will obtain the system of equation as

$$x_1^2 + x_2 = (11.1 - 10.9)\beta_1 + 10.9 = 0.2\beta_1 + 10.9 \tag{33}$$

$$x_1 + x_2^2 = (7.1 - 6.9)\beta_2 + 6.9 = 0.2\beta_2 + 6.9 \tag{34}$$

Now solving the above two equations (33) and (34) we will get

$$x_1^4 - (21.8 + 0.4\beta_1)x_1^2 + x_1 = 0.2\beta_2 - 4.36\beta_1 - 0.04\beta_1^2 - 111.91 \tag{35}$$

Now taking all the four combinations of β_1 and β_2, we will have four set of solutions. Out of those four, the minimum and maximum will be obtained as lower and upper bound of the solution.

So the solution of (32) in interval form is

$$x_1 = [2.9782,3.0217]$$

$$x_2 = [1.9693,2.0302]$$

The solution here obtained is same as the vertex method.

4.3.2 Example 7

Finally, we consider the equations as

$$\begin{gathered} x_1^2 + x_2 = [36.85,37.05] \\ x_1 + x_2^2 = [6.95,7.1] \\ x_1 + x_2 + x_3 = [10.9,11.15] \end{gathered} \tag{36}$$

Now converting the intervals into crisp, we will obtain the system of equation as

$$x_1^2 + x_2 = (37.05 - 36.85)\beta_1 + 36.85 = 0.2\beta_1 + 36.85 \tag{37}$$

$$x_1 + x_2^2 = (7.1 - 6.95)\beta_2 + 6.95 = 0.15\beta_2 + 6.95 \tag{38}$$

$$x_1 + x_2 + x_3 = (11.15 - 10.9)\beta_2 + 10.9 = 0.25\beta_3 + 10.9 \tag{39}$$

Now solving first two equations (37) (38) we will get

$$x_1^4 - (73.7 + 0.4\beta_1)x_1^2 + x_1 = 0.15\beta_2 - 14.74\beta_1 - 0.04\beta_1^2 - 1350.9725$$

Now taking all the four combinations of β_1 and β_2, we will have four set of solutions. Then by substituting the value of x_1 and x_2 in the equation (39) and considering the value of β_3 as 0 and 1, we will get eight set of solutions. Out of those eight, the minimum and maximum will be obtained as lower and upper bound of the solution.

So the solutions in interval form are found to be

$$x_1 = [5.9827, 6.0066]$$

$$x_2 = [0.9713, 1.0570]$$

$$x_2 = [3.8511, 4.1806]$$

Chapter 5

CONCLUSION

The main aim of this project has been to solve interval nonlinear system of equations. Vertex method is developed to solve interval nonlinear system of equations. The method is then verified by considering different types of examples. An application problem of nonlinear system of equations in electrical circuit analysis has also been discussed. This application problem has also been compared in crisp case with the available result and it was seen the crisp results contained in the presently obtained interval solution. So, presently proposed procedure may be used to solve nonlinear system of equations in interval form. It is known that in vertex method, as the number of unknowns increase the combination also increases exponentially. So for larger number of variables the vertex method may not be computationally efficient. Further, the main disadvantage of this method mentioned in [16] is that it assumes that the solution of the problem is a monotonic function of the interval parameters.

Next Krawczyk method has been developed to solve interval nonlinear system of equations without taking the combinations of lower and upper bounds as in Vertex method. Finally, a new method has been proposed by converting the interval into parametric form. The method has been validated by considering different type of examples. This method takes less time and less computation than vertex method. The methods can be easily extended to solve fully interval nonlinear system of equations.

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LIST OF PUBLICATIONS

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