k-BALANCING NUMBERS AND PELL'S EQUATION OF HIGHER ORDER

A report

submitted by

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of

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of

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DECLARATION

I here by declare that the topic "k-Balancing Numbers and Pell's Equation of Higher Order" submitted for the partial fulfillment of my M.Sc. has not been submitted any other institution or the university for the award of any other degree or diploma.

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CERTIFICATE

This is to certify that the project report entitled "*k*-**Balancing Numbers and Pell's Equation of Higher Order"** submitted by **Juli Sahu** to the National Institute, Rourkela, Odisha for the partial fulfillment of requirements for the degree of Master of Science in Mathematics is a bonafide record of research and review work carried out by her under my supervision. The contents of the project,in full or in parts have not been submitted to any other institution or the university or the award of any other Degree or Diploma.

11th May, 2015

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ABSTRACT

First time introduced in the year 1999, the balancing numbers are extensively studied. Each balancing number is associated with a Lucas-balancing number and are useful in the computation of balancing numbers of higher order. In this report, we study the sums of k-balancing numbers with indexes in an arithmetic sequence, say $an+r$ for fixed integers a and r . Also an infinite family of Pell's equations of degree $n \geq 2$ are discussed.

Contents

Chapter 0

Introduction

The concept of Fibonacci numbers was first discovered by the famous Italian mathematician Leonardo Fibonacci. The Fibonacci series was derived from the solution to a problem about rabbits.They can be obtained by the recursive formula [11, 12],

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 2,$$

with initial values $F_1 = 1,\ F_2 = 1$. Falcon and Plaza [9] studied k-Fibonacci numbers with indices in an arithmetic progression. For any integer number $k \geq 1$ the k^{th} Fibonacci sequences, say $\{F_{k,n}\}_{n \in N}$ is defined by the recurrence relation

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1,$$

with initial values $F_{k,0} = 0,\ F_{k,1} = 1$.

In the year 1999 Behera and Panda introduced the concept of balancing numbers. Balancing numbers n and balancers r are solutions of the Diophantine equation [3, 4, 8],

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r).$$

6, 35 and 204 are balancing numbers with balancer 2, 14 and 84 respectively.

They also proved that the recurrence relation for balancing numbers is

$$B_{n+1} = 6B_n - B_{n-1} \text{ for } n \geq 2,$$

where B_n is the n^{th} balancing number with $B_1 = 1$ and $B_2 = 6$. It has already been proved that n is a balancing number if and only if n^2 is a triangular number, that is $8n^2 + 1$ is a perfect square. If n is a balancing number, $C_n = \sqrt{8n^2 + 1}$ is called a Lucas-balancing number [3, 4, 8].

The recurrence relation for Lucas-balancing numbers is same as that of balancing numbers, i.e,

$$C_{n+1} = 6C_n - C_{n-1} \text{ for } n \geq 2,$$

where C_n is the n^{th} Lucas-balancing number with $C_1 = 3$ and $C_2 = 17$.

Another famous mathematician Liptai, later showed that the only balancing number in the sequence of Fibonacci numbers is 1. Moreover The closed form of both balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \ \ and \ \ C_n = \frac{\alpha^n + \beta^n}{2}.$$

The recurrence relation for balancing and Lucas-balancing are popularly known as Binets formulas for balancing and Lucas-balancing numbers $[3,8]$. This paper is a combination of three major results which uses a great deal of the above concepts and the resulting derivations.

Chapter 1

Preliminaries

In this chapter, include some known definitions which are frequently used in this work.

1.1 Recurrence Relation

A recurrence relation is an equation that defines a sequence recursively; where each term of the sequence is defined as a function of the preceding terms $[5,11]$.

1.2 Triangular Numbers

A number of the form $n(n+1)/2$ where $n \in Z^+$ is known as a triangular number.The triangular number $n(n+1)/2$ represents the area of a right angled triangle with base $n+1$ and perpendicular n. It is well known that $n \in Z^+$ is a "Triangular Number" if and only if $8n+1$ is a perfect square $[5,11,12]$.

1.3 Fibonacci Sequence

The Fibonacci sequence $[11,12]$ is defined recursively as $F_1 = 1,\ F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 2.$$

1.4 Lucas Sequence

The Lucas sequence $[11,12]$ is defined recursively as $L_1 = 1,\ L_2 = 3$ and

$$L_{n+1} = L_n + L_{n-1} \text{ for } n \geq 2.$$

1.5 Binet Formula

While solving a recurrence relation as a difference equation, the n^{th} term of the sequence is obtained in closed form, which is a formula containing conjugate surds of irrational numbers is known as the Binet formula for the particular sequence. These surds are obtained from the auxiliary equation of the recurrence relation for the recursive sequence under consideration [8,11,12]. Binet formula for Fibonacci and Lucas sequence are respectively

$$F_n = \frac{{\sigma_1}^n - {\sigma_2}^n}{\sigma_1 - \sigma_2} \ \ and \ \ L_n = {\sigma_1}^n + {\sigma_2}^n,$$

where $\sigma_1 = \frac{1+\sqrt{5}}{2}$ and $\sigma_2 = \frac{1-\sqrt{5}}{2}$.

1.6 Pell Sequence

The Pell sequence are defined recursively as [11,12],

$$P_{n+1} = 2P_n + P_{n-1} \ \text{ for } \ n \geq 2,$$

where $P_1 = 1$ and $P_2 = 2$.

1.7 Associated Pell Sequence

The associated Pell sequence is also determined from the same recurrence relation as that of Pell numbers as [11,12],

$$Q_{n+1} = 2Q_n + Q_{n-1} \ \text{ for } \ n \geq 2,$$

where $Q_1 = 1$ and $Q_2 = 3$.

1.8 Balancing Sequence

The solutions n and r of the Diophantine equation [3,4,8],

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$$

are called balancing numbers and balancers respectively.

The recurrence relation for balancing sequence is

$$B_{n+1} = 6B_n - B_{n-1} \ \text{ for } \ n \geq 2,$$

where B_n is the n^{th} balancing number with $B_1 = 1$ and $B_2 = 6$. n is a balancing number if and only if n^2 is a triangular number, that is $8n^2 + 1$ is a perfect square.

1.9 Lucas-balancing Sequence

If n is a balancing number, $C_n = \sqrt{8n^2 + 1}$ is called a Lucas-balancing number [3,4,8].
The recurrence relation for Lucas-balancing sequence is

$$C_{n+1} = 6C_n - C_{n-1} \ \text{ for } \ n \geq 2,$$

where C_n is the n^{th} Lucas-balancing number with $C_1 = 3$ and $C_2 = 17$.

Chapter 2

On k-balancing numbers

2.1 Introduction

The Fibonacci numbers are defined by the recurrence relation [11, 12],

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 2, \tag{2.1.1}$$

where F_n is the n^{th} Fibonacci number with $F_1 = 1$ and $F_2 = 1$. In [9], Falcon and Plaza generalized the definition of Fibonacci numbers and defined k-Fibonacci numbers as follows:

Definition 2.1.1. For any integer number $k \geq 1$ the k^{th} Fibonacci sequences, say $\{F_{k,n}\}_{n\in N}$ is defined by the recurrence relation [9],

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1, \tag{2.1.2}$$

where $F_{k,0} = 0$, and $F_{k,1} = 1$.

When $k = 1$ the k^{th} Fibonacci sequence reduces to Fibonacci sequence. while for $k = 2$, it reduces to Pell's sequence. In [9], Falcon and Plaza studied k-Fibonacci numbers with indices in an arithmetic progression. In this chapter, we define k^{th} balancing numbers and consider the sums of k-balancing numbers with indices in an arithmetic sequence, say $an + r$ for fixed integers a and r. This enables us to give several formulas for the sums of such numbers.

Definition 2.1.2. For any integer number $k \geq 1$, we define the k^{th} Balancing sequence as say $\{B_{k,n}\}_{n\in N}$ is defined by the recurrence relation

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1} \text{ for } n \geq 1, \tag{2.1.3}$$

where $B_{k,0} = 0, \quad B_{k,1} = 1$.

2.2 Some properties of the k-Fibonacci and k-balancing numbers

The Binet formula for k-Fibonacci numbers [9] is

$$F_{k,n} = \frac{\sigma_1{}^n - \sigma_2{}^n}{\sigma_1 - \sigma_2},$$

where $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$.

The following important formulas available in [9], can be proved using the Binet formula. They are needed to prove results of the subsequent sections. For $n,\ m \geq 0$.

1. Catalan's identity: $F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1+r}F_{k,r}^2$.
2. Simson's identity: $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$.
3. D'Ocagne's identity: $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}$.
4. Convolution product: $F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1} = F_{k,n+m}$.
5. For all integers $n \geq 1$, $\alpha^n + \beta^n = F_{k,n+1} + F_{k,n-1}$.
6. $F_{k,a(n+2)+r} = (F_{k,a-1} + F_{k,a+1})F_{k,a(n+1)+r} - (-1)^a F_{k,an+r}$.

It is easy to see that the Binet formula for k-balancing numbers is

$$B_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = 3k + \sqrt{9k^2-1}$ and $\beta = 3k - \sqrt{9k^2-1}$.

For $n,\ m \geq 0$, by using The Binet formula, we can easily get the following new formulas.

1. Catalan's identity: $B_{k,n}^2 - B_{k,n-r}B_{k,n+r} = B_{k,r}^2$.
2. Simson's identity: $B_{k,n}^2 - B_{k,n-1}B_{k,n+1} = 1$.
3. D'Ocagne's identity: $B_{k,m}B_{k,n+1} - B_{k,m+1}B_{k,n} = B_{k,m-n}^2$.
4. Convolution product: $B_{k,n+1}B_{k,m} - B_{k,n}B_{k,m-1} = B_{k,n+m}$.

2.3 On the k-balancing numbers with indices of the form $an + r$

Lemma 2.3.1. *For a given natural number* $n(n \geq 1)$,

$$\alpha^n + \beta^n = B_{k,n+1} - B_{k,n-1}. \tag{2.3.1}$$

Proof. By applying Binet formula and taking $\alpha\beta = 1$.

$$\begin{aligned} B_{k,n+1} - B_{k,n-1} &= \frac{\alpha^{n+1} - \beta^{n+1} - \alpha^{n-1} + \beta^{n-1}}{\alpha - \beta} \\ &= \frac{\alpha^n(\alpha - \alpha^{-1}) - \beta^n(\beta - \beta^{-1})}{\alpha - \beta} \\ &= \frac{\alpha^n(\alpha - \beta) + \beta^n(\alpha - \beta)}{\alpha - \beta} \\ &= \alpha^n + \beta^n. \end{aligned}$$

□

Lemma 2.3.2. *For natural numbers n and r,*

$$B_{k,a(n+2)+r} = (B_{k,a+1} - B_{k,a-1})B_{k,a(n+1)+r} - B_{k,an+r}. \tag{2.3.2}$$

Proof. By using the above Lemma 2.3.1 and Binet formula we can prove;

$$\begin{aligned}(B_{k,a+1} - B_{k,a-1})B_{k,a(n+1)+r} &= (\alpha^a + \beta^a)\frac{(\alpha^{an+a+r} - \beta^{an+a+r})}{\alpha - \beta} \\ &= \frac{\alpha^{a(n+2)+r} - \beta^{a(n+2)+r} + \alpha^{an+r} - \beta^{an+r}}{\alpha - \beta} \\ &= B_{k,a(n+2)+r} + B_{k,an+r}.\end{aligned}$$

□

2.4 Generating function of the sequence $\{B_{k,an+r}\}$

Let $f_{a,r}(k,x)$ be the generating function of the Fibonacci sequence $\{F_{k,an+r}\}$ with $0 \le r \le a-1$, Then the following result was proved in [9]

$$f_{a,r}(k,x) = \frac{F_{k,r} + (-1)^r F_{k,a-r}x}{1 - L_{k,a}x + (-1)^a x^2}. \tag{2.4.1}$$

Theorem 2.4.1. *The generating function $g_{a,r}(k,x)$ of the sequence $\{B_{k,an+r}\}$ with $0 \le r \le a-1$, is*

$$f_{a,r}(k,x) = \frac{B_{k,r} - B_{k,r-a}x}{1 - 2C_{k,a}x + x^2}. \tag{2.4.2}$$

Proof.

$$\begin{aligned}&(1 - 2C_{k,a}x + x^2)f_{a,r}(k,x) \\ &= (1 - 2C_{k,a}x + x^2)(B_{k,r} + B_{k,a+r}x + B_{k,2a+r}x^2 + \ldots) \\ &= B_{k,r} + (B_{k,a+r} - 2C_{k,a}B_{k,r})x + (B_{k,2a+r} - 2C_{k,a}B_{k,a+r} + B_{k,r})x^2 \\ &\qquad + (B_{k,3a+r} - 2C_{k,a}B_{k,2a+r} + B_{k,a+r})x^3 + \ldots. \\ &= B_{k,r} + (B_{k,a+r} - 2C_{k,a}B_{k,r})x + \left[\sum_{n\ge 2} B_{k,a(n+2)+r} - 2C_{k,a}B_{k,a(n+1)+r} + B_{k,an+r}\right] \\ &= B_{k,r} + (B_{k,a+r} - 2C_{k,a}B_{k,r})x + \left[\sum_{n\ge 2} -B_{k,an+r} + B_{k,an+r}\right] \\ &= B_{k,r} + (B_{k,a+r} - 2C_{k,a}B_{k,r})x \\ &= B_{k,r} - B_{k,r-a}x.\end{aligned}$$

Hence, the generating function for the initial power series is

$$f_{a,r}(k,x) = \frac{B_{k,r} - B_{k,r-a}x}{1 - 2C_{k,a}x + x^2}.$$

□

2.4.1 Particular cases

The generating functions of the sequences $\{B_{k,an+r}\}$ for different values of the parameter a *and* r are

(1) $a = 1$ and $r = 0 : f_{1,0}(k,x) = \frac{x}{1-6kx+x^2}$.

(2) $a = 2 : (i) r = 0 : f_{2,0}(k,x) = \frac{6kx}{1-36k^2x+2x+x^2}$. $(ii) r = 1 : f_{2,1}(k,x) = \frac{1+x}{1-2x(18k^2-1)+x^2}$.

(3) $a = 3 : (i) r = 0 : f_{3,0}(k,x) = \frac{36k^2x-x}{1-2x(108k^3-9k)+x^2}$. $(ii) r = 1 : f_{3,1}(k,x) = \frac{1+6kx}{1-2x(108k^3-9k)+x^2}$.

2.5 Sum of k-balancing numbers of kind $an+r$

The following Fibonacci results are available in [9],

$$\sum_{i=0}^{n} F_{k,ai+r} = \frac{F_{k,a(n+1)+r} - (-1)^a F_{k,an+r} - (-1)^r F_{k,a-r} - F_{k,r}}{F_{k,a+1} + F_{k,a-1} - (-1)^a - 1}. \tag{2.5.1}$$

Theorem 2.5.1. *Sum of k-balancing numbers of kind* $an+r$

$$\sum_{i=0}^{n} B_{k,ai+r} = \frac{B_{k,an+r} - B_{k,a(n+1)+r} + B_{k,a-r} + B_{k,r}}{2 - B_{k,a+1} + B_{k,a-1}}. \tag{2.5.2}$$

Proof. By using Binet formula, we get

$$\begin{aligned}
\sum_{i=0}^{n} B_{k,ai+r} &= \sum_{i=0}^{n} \frac{\alpha^{ai+r} - \beta^{ai+r}}{\alpha - \beta} \\
&= \frac{1}{\alpha - \beta}\left[\sum_{i=0}^{n} \alpha^{ai+r} - \sum_{i=0}^{n} \beta^{ai+r}\right] \\
&= \frac{1}{\alpha - \beta}\left[\frac{\alpha^{an+r+a} - \alpha^r}{\alpha^a - 1} - \frac{\beta^{an+r+a} - \beta^r}{\beta^a - 1}\right] \\
&= \frac{1}{\alpha - \beta}\left[\frac{\alpha^{an+r} - \beta^{an+r} - \alpha^{an+r+a} + \beta^{an+r+a} + \alpha^{a-r} - \beta^{a-r} + \alpha^r - \beta^r}{2 - (\alpha^a + \beta^a)}\right] \\
&= \frac{B_{k,an+r} - B_{k,a(n+1)+r} + B_{k,a-r} + B_{k,r}}{2 - B_{k,a+1} + B_{k,a-1}}.
\end{aligned}$$

□

2.6 Particular cases

2.6.1 Case-1

Sum of odd k-balancing numbers, If $a = 2p+1$ then Eq.(2.5.2) is

$$\sum_{i=0}^{n} B_{k,(2p+1)i+r} = \frac{B_{k,(2p+1)n+r} - B_{k,(2p+1)(n+1)+r} + B_{k,(2p+1)-r} + B_{k,r}}{2 - B_{k,2p+2} + B_{k,2p}}. \tag{2.6.1}$$

Examples

(1) If $p=0$ then $a=1, r=0$ and $\sum_{i=0}^{n} B_{k,i} = \frac{B_{k,n}-B_{k,(n+1)}+B_{k,1}}{2-B_{k,2}} = \frac{B_{k,n}-B_{k,(n+1)}+1}{2-6k}$.

(a) For $k=1$, the formula for classical balancing sequence,

$$\sum_{i=0}^{n} B_i = \frac{B_n - B_{n+1} + 1}{2-6k} = \frac{B_{n+1} - B_n - 1}{4}.$$

(b) For $k=2$, the formula for the Pell's sequence is $\sum_{i=0}^{n} P_i = \frac{P_n - P_{n+1}+1}{2-6k} = \frac{P_{n+1}-P_n-1}{10}$.

(2) If $p=1$ and $a=3$, then $\sum_{i=0}^{n} B_{k,3i+r} = \frac{B_{k,3n+r}-B_{k,3(n+1)+r}+B_{k,3-r}+B_{k,r}}{2-216k^3+18k}$.

$(a) r=0: \sum_{i=0}^{n} B_{k,3i} = \frac{B_{k,3n}-B_{k,3(n+1)}+36k^2-1}{2-216k^3+18k}$; for $k=1$,

the formula for classical balancing sequence is $\sum_{i=0}^{n} B_{3i} = \frac{B_{3n}-B_{3n+3}+35}{196}$.

$(b) r=1: \sum_{i=0}^{n} B_{k,3i+1} = \frac{B_{k,3n+1}-B_{k,3n+4}+6k+1}{2-216k^3+18k}$; for $k=1$,

the formula for classical balancing sequence is $\sum_{i=0}^{n} B_{3i+1} = \frac{B_{3n+1}-B_{3n+4}+7}{196}$.

$(c) r=2: \sum_{i=0}^{n} B_{k,3i+2} = \frac{B_{k,3n+2}-B_{k,3n+5}+6k+1}{2-216k^3+18k};\ fork=1,$

the formula for classical balancing sequence is $\sum_{i=0}^{n} B_{3i+2} = \frac{B_{3n+2}-B_{3n+5}+7}{196}$.

(3) If $p=2$ and $a=5$, then $\sum_{i=0}^{n} B_{k,5i+r} = \frac{B_{k,5n+r}-B_{k,5(n+1)+r}+B_{k,5-r}+B_{k,r}}{2-7776k^5+1080k^3-30k}$.

$(a) r=0: \sum_{i=0}^{n} B_{k,5i} = \frac{B_{k,5n}-B_{k,5n+5}+B_{k,5}}{2-7776k^5+1080k^3-30k}$.

2.6.2 Case-2

Sum formula for even k-balancing numbers: If $a=2p$ then Eq.$(2.5.2)$ is

$$\sum_{i=0}^{n} B_{k,2pi+r} = \frac{B_{k,2pn+r} - B_{k,2p(n+1)+r} + B_{k,2p-r} + B_{k,r}}{2 - B_{k,2p+1} + B_{k,2p-1}}. \tag{2.6.2}$$

For Example :

(1) If $p=1$ then $a=2$ then $\sum_{i=0}^{n} B_{k,2i+r} = \frac{B_{k,2n+r}-B_{k,2(n+1)+r}+B_{k,2-r}+B_{k,r}}{4-36k^2}$.

$(a) r=0: \sum_{i=0}^{n} B_{k,2i} = \frac{B_{k,2n}-B_{k,2(n+1)+6k}}{4-36k^2};\ for\ k=1,$

the formula for classical balancing sequence is $\sum_{i=0}^{n} B_{2i} = \frac{B_{2n+2}-B_{2n}-6}{32}$.

$(b) r=1: \sum_{i=0}^{n} B_{k,2i+1} = \frac{B_{k,2n+1}-B_{k,2n+3}+2}{4-36k^2};\ for\ k=1,$

the formula for classical balancing sequence is $\sum_{i=0}^{n} B_{2i+1} = \frac{B_{2n+3}-B_{2n+1}-2}{32}$.

(2) If $p=2$ then $a=4$ then $\sum_{i=0}^{n} B_{k,4i+r} = \frac{B_{k,4n+r}-B_{k,4(n+1)+r}+B_{k,4-r}+B_{k,r}}{144k^2-1296k^4}$.

$(a) r=0 : \sum_{i=0}^{n} B_{k,4i} = \frac{B_{k,4n}-B_{k,4n+4}+216k^3-12k}{144k^2-1296k^4}$.

$(b) r=1 : \sum_{i=0}^{n} B_{k,4i+1} = \frac{B_{k,4n+1}-B_{k,4n+5}-36k^2+2}{144k^2-1296k^4}$.

The following Fibonacci results are available in [9].The sum of k-Fibonacci numbers of order $an+r$ is

$$\sum_{i=0}^{n}(-1)^i F_{k,ai+r} = \frac{(-1)^{n+a}F_{k,an+r}+(-1)^n F_{k,a(n+1)+r}+(-1)^{r+1}F_{k,a-r}+F_{k,r}}{F_{k,a+1}+F_{k,a-1}+(-1)^a+1}. \tag{2.6.3}$$

Theorem 2.6.1. *The alternating sum of k-balancing numbers of order $an+r$ is*

$$\sum_{i=0}^{n}(-1)^i B_{k,ai+r} = \frac{(-1)^{an+2a}B_{k,an+r}+(-1)^{an+2a}B_{k,a(n+1)+r}+(-1)^a B_{k,a-r}+B_{k,r}}{2+(-1)^{a+1}(B_{k,a+1}-B_{k,a-1})}. \tag{2.6.4}$$

Proof. Applying Binet formula, we get

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i B_{k,ai+r} &= (-1)^i \sum_{i=0}^{n} \frac{\alpha^{ai+r}-\beta^{ai+r}}{\alpha-\beta} \\
&= \frac{(-1)^i}{\alpha-\beta}\left[\sum_{i=0}^{n}\alpha^{ai+r} - \sum_{i=0}^{n}\beta^{ai+r}\right] \\
&= \frac{1}{\alpha-\beta}\left[\frac{(-1)^{an+a}\alpha^{an+r+a}-\alpha^r}{(-1)^a\alpha^a-1} - \frac{(-1)^{an+a}\beta^{an+r+a}-\beta^r}{(-1)^a\beta^a-1}\right] \\
&= \frac{(-1)^{an+2a}B_{k,an+r}+(-1)^{an+2a}B_{k,a(n+1)+r}+(-1)^a B_{k,a-r}+B_{k,r}}{2+(-1)^{a+1}(B_{k,a+1}-B_{k,a-1})}.
\end{aligned}$$

□

For Example:

(1) $a=1$ and $r=0$ then, $\sum_{i=0}^{n}(-1)^i B_{k,i} = \frac{(-1)^{n+2}B_{k,n}+(-1)^{n+2}B_{k,n+1}-1}{2+6k}$.

(2) $a=2$ and $r=0$ then, $\sum_{i=0}^{n}(-1)^i B_{k,2i} = \frac{B_{k,2n}+B_{k,2n+2}+6k}{4-36k^2}$.

(3) $a=2$ and $r=1$ then, $\sum_{i=0}^{n}(-1)^i B_{k,2i+1} = \frac{B_{k,2n+1}+B_{k,2n+3}+2}{4-36k^2}$.

(4) $a=4$ and $r=0$ then, $\sum_{i=0}^{n}(-1)^i B_{k,4i} = \frac{B_{k,4n}+B_{k,4n+4}+(216k^3-12k)}{144k^2-1296k^4}$.

(5) $a=4$ and $r=1$ then, $\sum_{i=0}^{n}(-1)^i B_{k,4i+1} = \frac{B_{k,4n+1}+B_{k,4n+5}+36k^2}{144k^2-1296k^4}$.

Chapter 3

Generalized balancing number and Pell's equations of higher degree

3.1 Introduction

Let d be a non-square positive integer. The Diophantine equation $x^2 - dy^2 = 1$ is called the Pell's equation. It is well known that given the smallest positive solution (x_0, y_0), all solutions (x_n, y_n) can be obtained from [11, 12]

$$y_n + \sqrt{d}x_n = (x_0 + \sqrt{d}y_0)^n. \tag{3.1.1}$$

Observe that the Pell's equation can be written in the form

$$\det \begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1. \tag{3.1.2}$$

In a recent paper, [2] has generalized Pell's equation to higher degree as follows.

Let A and B be non-zero integers then the second order linear recursive sequences $R = \{R_n\}_{n=0}^{\infty}$ and $V = \{V_n\}_{n=0}^{\infty}$ are defined by the recursions

$$R_n = AR_{n-1} + BR_{n-2} \text{ and } V_n = AV_{n-1} + BV_{n-2} \text{ for } n \geq 2, \tag{3.1.3}$$

$R_0 = 0,\ R_1 = 1,\ V_0 = 2$ and $V_1 = A$. If $A = B = 1$ then $R_n = F_n$ and $V_n = L_n$, where F_n and L_n denotes the nth Fibonacci and Lucas numbers respectively.

We define a generalized balancing sequence by the recursions

$$R_n = 6AR_{n-1} - BR_{n-2} \text{ and } V_n = 6AV_{n-1} - BV_{n-2} \text{ for } n \geq 2, \tag{3.1.4}$$

while $R_0 = 0,\ R_1 = 1,\ V_0 = 1$ and $V_1 = 3A$. If $A = B = 1$ then $R_n = B_n$ and $V_n = C_n$, where B_n and C_n denotes the nth balancing and Lucas balancing numbers respectively.

The polynomial $g(x) = x^2 - 6Ax + B$ is said to be the characteristic polynomial of the sequences R and V. The complex numbers α and β are the roots of $g(x) = 0$.
Then, the Binet's formula are

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \frac{\alpha^n + \beta^n}{2} \text{ for } n \geq 0. \tag{3.1.5}$$

The Pell's equation $x^2 - dy^2 = \pm 1 (d \in Z)$ can be written as

$$\det \begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1.$$

The quasi-cyclic matrix is defined in [6, 7, 10] as

$$Q_n = Q_n(d; x_1, x_2, x_3, \ldots, x_n) = \begin{pmatrix} x_1 & dx_n & dx_{n-1} & \cdots & dx_2 \\ x_2 & x_1 & dx_n & \cdots & dx_3 \\ x_3 & x_2 & x_1 & \cdots & dx_4 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ x_n & x_{n-1} & x_{n-2} & \ldots & x_1 \end{pmatrix}, \tag{3.1.6}$$

i.e, every entry of the upper triangular part(not including the main diagonal) of the cyclic matrix of entries $x_1, x_2, x_3, \ldots, x_n$ is multiplied by d. The equation

$$\det(Q_n) = \pm 1,$$

i.e,

$$det \begin{pmatrix} x_1 & dx_n & dx_{n-1} & \cdots & dx_2 \\ x_2 & x_1 & dx_n & \cdots & dx_3 \\ x_3 & x_2 & x_1 & \cdots & dx_4 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ x_n & x_{n-1} & x_{n-2} & \ldots & x_1 \end{pmatrix} = \pm 1, \tag{3.1.7}$$

is called Pell's equation of degree $n \geq 2$.

If $n = 3$ then (3.1.7) has the form

$$x_1^3 + dx_2^3 + d^2 x_3^3 - 3dx_1x_2x_3 = \pm 1.$$

3.2 The main results and their proofs

In a recent paper [6], for $n \geq 2$,

$$\det(Q_n(L_n; F_{2n-1}F_{2n-2}, \ldots, F_n) = 1 \quad \text{for} \quad n \geq 2, \tag{3.2.1}$$

where F_n and L_n denotes Fibonacci and Lucas numbers respectively.

We consider a similar result for generalized balancing numbers.

Theorem 3.2.1. *Let* $n \geq 2$*, then*

$$\det(Q_n(2C_n; B_{2n-1}B_{2n-2}, \ldots, B_n) = 1, \tag{3.2.2}$$

where B_n *and* C_n *denotes balancing and Lucas balancing numbers respectively.*

Proof. For $n = 2$, we have

$$\det(Q_n(2C_2;B_3,B_2)) = det\begin{pmatrix} 35 & 204 \\ 6 & 35 \end{pmatrix} = 1.$$

So, The result is true for n=2. We have to prove this result for $n > 2$, let

$$T = \begin{pmatrix} 1 & -6 & 1 & \dots & 0 & 0 \\ 0 & 1 & -6 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -6 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \qquad (3.2.3)$$

By multiplication of matrices and properties of balancing and Lucas balancing, we have

$$Q_nT = \begin{pmatrix} B_{2n-1} & -B_{2n-2} & 1 & 0 & \dots & 0 \\ B_{2n-2} & -B_{2n-3} & 0 & 1 & \dots & 0 \\ B_{2n-3} & -B_{2n-4} & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & 1 \\ B_{n+1} & -B_n & 0 & 0 & \dots & 0 \\ B_n & -B_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}. \qquad (3.2.4)$$

Taking the determinant of both sides of (3.2.4) and det(T)=1, we have

$$\begin{aligned} \det(Q_n) &= \det(Q_n)\det(T) = \det(Q_nT) \\ &= (-1)^{2n-4} det\begin{pmatrix} B_{n+1} & -B_n & 0 & 0 & \dots & 0 \\ B_n & -B_{n-1} & 0 & 0 & \dots & 0 \\ B_{2n-1} & -B_{2n-2} & & & & \\ B_{2n-2} & -B_{2n-3} & & & & \\ \vdots & \vdots & & & I_{n-2} & \\ B_{n+2} & -B_{n+1} & & & & \end{pmatrix} \\ &= \det\begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}\det(I_{n-2}) \\ &= (B_n^2 - B_{n+1}B_{n-1}) \\ &= 1. \end{aligned}$$

Thus, Theorem 3.2.1 is true. □

Lemma 3.2.2. *Let the sequences R and V be defined by* (3.1.4) *and* $\alpha \neq \beta$ *in* (3.1.5), *then*

1. $R_{n+1}R_{n-1} - R_n^2 = -B^n (n \geq 1)$,
2. $2V_nR_n = R_{2n} (n \geq 0)$,
3. $2V_nR_{n+1} = R_{2n+1} + B (n \geq 0)$,

4. $E_n^n = 2V_nI_n$ *and* $E_n^{n+1} = 2V_nE_n (n \geq 3)$,

where E_n *is defined as*

$$E_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 2V_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots\dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{3.2.5}$$

Proof. The first three properties of the Lemma are known or using (3.1.5), they can be prove easily.For the proof of (4) Lemma (3.2.2) of consider the multiplication of matrices,

$$E_n^2 = E_n.E_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 2V_n & 0 \\ 0 & 0 & \dots & 0 & 0 & 2V_n \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{pmatrix},$$

$$E_n^3 = E_n^2.E_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 2V_n & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 2V_n & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 2V_n \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_n^n = \begin{pmatrix} 2V_n & 0 & \dots & 0 & 0 \\ 0 & 2V_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2V_n & 0 \\ 0 & 0 & \dots & 0 & 2V_n \end{pmatrix} = 2V_nI_n,$$

hence, $E_n^{n+1} = E_n^n.E_n = (2V_nI_n)E_n = 2V_nE_n$. □

Theorem 3.2.3. *By using* [2] *for* $n \geq 2$ *we can show that*

$$\det(Q_n(V_n; R_{2n-1}, R_{2n-2}, \dots, R_n)) = B^{n(n-1)},$$

i.e, $(x_1, x_2, x_3, \dots, x_n) = (R_{2n-1}, R_{2n-2}, \dots, R_n)$ *is a solution of the generalized pell's equation of degree n,*

$$\det(Q_n(V_n; x_1, x_2, x_3, \dots, x_n)) = B^{n(n-1)}.$$

Proof. For $n = 2$ we get that

$$\det(Q_2(2V_2; R_3, R_2)) = \begin{vmatrix} 36A^2 - B & 6A(36A^2 - 2B) \\ 6A & 36A^2 - B \end{vmatrix} = B^2.$$

If $n > 2$, let us consider the $n \times n$ matrices

$$T_n = \begin{pmatrix} 1 & -6A & B & \dots & 0 & 0 \\ 0 & 1 & -6A & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -6A & B \\ 0 & 0 & 0 & \dots & 1 & -6A \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

$$Q_n = (Q_n(2V_n;R_{2n-1},R_{2n-2},\dots,R_n)) = \begin{pmatrix} R_{2n-1} & 2V_nR_n & \dots & 2V_nR_{2n-2} \\ R_{2n-2} & R_{2n-1} & \dots & 2V_nR_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ R_n & R_{n+1} & \dots & R_{2n-1} \end{pmatrix}.$$

Then, by (3.1.4),(3.1.5) and (1)-(3) of Lemma 3.2.2, we can verify that

$$Q_n.T_n = \begin{pmatrix} R_{2n-1} & -BR_{2n-2} & B & 0 & \dots & 0 \\ R_{2n-2} & -BR_{2n-3} & 0 & B & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{n+2} & -BR_{n+1} & 0 & 0 & \dots & B \\ R_{n+1} & -BR_n & 0 & 0 & \dots & 0 \\ R_n & -BR_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Taking the determinant of $Q_n.T_n$ we get that

$$\begin{aligned} \det(Q_n.T_n) &= (-1)^{2n+2}(BR_n^2 - BR_{n+1}R_{n-1})\det(BI_{n-2}) \\ &= (BR_n^2 - BR_{n+1}R_{n-1})B^{n-2} \\ &= B^{n-1}(R_n^2 - R_{n-1}R_{n+1}) \\ &= B^{n(n-1)}. \end{aligned}$$

Hence, Theorem 3.2.3 proved. □

From [2] the inverse matrix is defined as:

$$Q_n^{-1}(V_n;R_{2n-1},R_{2n-2},\dots,R_n) = (-1)^{n-1}B^{-n}(BI_n + AE_n - E_n^2) \ \ for\ n \geq 3.$$

Theorem 3.2.4. *By using the above result we can show that for* $n \geq 3$*, the matrix* $Q_n(2V_n;R_{2n-1},R_{2n-2},\dots,R_n)$ *is invertible and it's inverse matrix* Q_n^{-1} *is as follows,*

$$Q_n^{-1}(2V_n;R_{2n-1},R_{2n-2},\dots,R_n) = (-1)B^{-2}(BI_n - 6AE_n + E_n^2),$$

where I_n *and* E_n *denotes the identity matrix of order n and* E_n *is defined by* (3.2.5).

Proof. Theorem 3.2.3 implies that

$Q_n^{-1}(2V_n;R_{2n-1},R_{2n-2},\dots,R_n)$ exists. It is easily verify that

$$Q_n(2V_n;R_{2n-1},R_{2n-2},\dots,R_n) = R_{2n-1}I_n + R_{2n-2}E_n + R_{2n-3}E_n^2 + \dots + R_nE_n^{n-1},$$

therefore we have to show that

$$3Q_n(-1)B^{-2}(BI_n - 6AE_n + E_n^2) = I_n,$$

i.e,

$$(R_{2n-1}I_n + R_{2n-2}E_n + R_{2n-3}E_n^2 + \ldots + R_nE_n^{n-1})(-1)B^{-2}(BI_n - 6AE_n + E_n^2) = I_n. \qquad (3.2.6)$$

By (3.1.4), the left hand side of (3.2.6) can be written as

$$(-1)B^{-2}(BR_{2n-1}I_n + BR_{2n-2}E_n - 6AR_{2n-1}E_n - 6AR_nE_n^n + R_{n+1}E_n^n + R_nE_n^{n+1} + O_n + \ldots + O_n), \qquad (3.2.7)$$

where O_n is the zero-matrix of order n.
Thus applying (3.1.4),(1)-(4)of Lemma 3.2.2 and (3.1.5), then (3.2.7) is equal to

$$\begin{aligned}
&(-1)B^{-2}(BR_{2n-1}I_n + (BR_{2n-2} - 6AR_{2n-1})E_n - 12AR_nV_nI_n + 2R_{n+1}V_nI_n + 2R_nV_nE_n)\\
&= (-1)B^{-2}(BR_{2n-1}I_n - 2B_{n-1}V_nI_n)\\
&= (-1)B^{-2}BI_n(R_{2n-1} - 2R_{n-1}V_n)\\
&= (-1)B^{-1}(-B)I_n.\\
&= I_n.
\end{aligned}$$

Hence, Theorem 3.2.4 is proved. □

Corollary 3.2.5.

$$(x_1, x_2, \ldots, x_n) = \begin{cases} (-1, 6A, -1, 0, \ldots, 0), & if\ B = 1, \\ (1, 6A, -1, 0, \ldots, 0), & if\ B = -1, \end{cases}$$

is an other solution of the generalized Pell's equation

$$\det(Q_n(2V_n; x_1, x_2, \ldots, x_n)) = 1. \qquad (3.2.8)$$

Proof. By Theorem 3.2.4,

$$\det(Q_n(2V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)).\det(Q_n^{-1}(2V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)) = 1,$$

thus, if $|B| = 1$ then by Theorem 3.2.3,

$$\det(Q_n^{-1}(2V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)) = 1.$$

Example:

Let B=1,then by Theorem 3.2.4,

$$
\begin{aligned}
Q_n^{-1}(2V_n;R_{2n-1},R_{2n-2},\ldots,R_n) &= -I_n+6AE_n-E_n^2 \\
&= \begin{pmatrix} -1 & 0 & \ldots & -2Q_n & 12AQ_n \\ 6A & -1 & \ldots & 0 & -2Q_n \\ -1 & 6A & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 6A & -1 \end{pmatrix},
\end{aligned}
$$

i.e, $(x_1,x_2,\ldots,x_n)=(-1,6A,-1,0,\ldots,0)$ is a solution of (3.2.8).

□

Bibliography

[1] A. Behera and G. K. Panda, *On the square roots of triangular numbers,* Fib.Quart., 37 (1999), 98-105.

[2] F. Matyas, *Second order linear recurrences and Pell's equations of higher degree ,* Acta Acad. Paed. Agriensis, Sectio Mathematicae 29 (2002), 47-53.

[3] G.K. Panda, *Some fascinating prop erties of balancing numbers, In Pro c. of Eleventh Internat,* Conference on Fib onacci Numb ers and Their Applications, Cong. Numerantium, 194, (2009), 185-189.

[4] G. K. Panda, *Sequence balancing and cobalancing numbers,* The Fibonacci Quarterly, 45(2007), 265-271.

[5] Hua Logeeg and Wang Yuan, *Applications of Number Theory to Numerical Analysis,* Science Publishing House (in Chinese), 1978; New York: Springer-Verlag, 1981.

[6] Lin Dazheng, *Fibonacci-Lucas Quasi-Cyclic Matrices,* The Fibonacci Quar-terly 40.2 (2002), 280-286.

[7] Lin Dazheng, *Fibonacci Matrices,* The Fibonacci Quarterly 37.2 (1999): 14-20. MR, 99m: 11011.

[8] P.K. Ray, *Balancing and cobalancing numbers,* Ph.D. Thesis, National Institute of Technology, Rourkela, (2009).

[9] Sergio Falcon, Angel Plaza, *On k-Fibonacci numbers of arithmetic indexes,* Applied Mathematics and Computation 208 (2009), 180-185.

[10] Shen Guangxing, *On Eigenvalues of Some Cyclic Matrices,* Math. Application 4.3 (1991), 76-82.

[11] Thomas Koshy, *Elementary Number Theory with Application,* Elsevier.

[12] T. Koshy, *Fibonacci and Lucas numbers with applications,* Pure and Applied Mathematics, Wiley-interscience, New York, (2001).