Master's Thesis

A Project Report on

Inflationary Cosmology: Scalar Field Models and Structure Formation

Submitted for the Partial Fulfillment of the Requirements for the Degree of

Integrated 5 Year Master of Science in Physics

By

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Abstract

Cosmology is the study of the universe as a whole. It deals with the origin and evolution of the universe. Our current understanding of our universe is based on the application of theory of general relativity to the universe. The standard theory of cosmology is able to explain many of the observational facts about the universe with great success, such as the expansion of the universe, primordial nucleosynthesis, origin and spectrum of Cosmic Microwave Background Radiation(CMBR) etc.

Despite of its great success, it remains silent about some of the most profound questions related to the initial conditions of our universe. There are problems such as the horizon problem and the flatness problem for which the standard theory of cosmology offers no solution. The solution to these problems is provided by the theory of inflation. In addition to solving the above problems, Inflation, as a bonus, provides seed for structure formation in the universe.

The theory of Inflation was proposed to solve the problems associated with the standard (FRW) cosmology. Inflation postulates the existence of a field ϕ, the inflaton, associated with which was the potential $V(\phi)$. The field was slowly rolling for some duration from its local minima to global minima. During this small duration of slow roll, ϕ was the dominant form of energy. The space experienced a period of exponential expansion. We have learned a great deal about inflation, but, we don't know much about the potential $V(\phi)$ itself. There exist hundreds of models of inflaton, with different $V(\phi)$s. The predictions of any successful model should match with the observational/experimental values of the parameters involved.

Here we study few such models to find out if they can be the potential inflaton. Also we explore the basic ideas about structure formation due to the quantum fluctuations in inflaton.

Certificate

This is to certify that the work done in this thesis entitled, "Inflationary Cosmology: Scalar Field Models and Structure Formation", is submitted by Abinas Pradhan towards the partial fulfillment of the requirements for the award of Integrated 5 Year Master of Science in Physics degree by National Institute of Technology Rourkela. The work presented here is reproduced by study and analysis of works done previously. It is a record of the work done by him, under my supervision.

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Chapter 1

A Quick Review of Cosmology

1.1 Observational Cosmology

Observational cosmology is the observational study of the universe. This includes observing the cosmic structures, measuring the dynamical parameters, and putting constraints on theoretical models.

The large scale observations of the sky have helped to reveal many astonishing facts about the universe we live in; such as (Ref. [20])

- **The Cosmological Principle**

  "*Viewed on a sufficiently large scale, the universe is* ***Homogeneous*** *and* ***Isotropic***."

  1. **Homogeneous**
     Homogeneous means location independent. The universe on a large scale appears to be homogeneous. By that we mean, the universe, on a sufficiently large scale, looks the same, independent of the position of the observer. The statistical distribution of matter-energy and cosmic structures (galaxies) in our universe appears to be the same, independent of the point of observation. The fact that our universe is homogeneous on a large scale is supported by deep sky surveys.

  2. **Isotropic**
     Isotropic means direction independent. There is no preferred direction in our universe. The universe, looks alike whichever direction we see. Isotropy is strongly supported by the observation and analysis of the **C**osmic **M**icrowave **B**ackground **R**adiation (CMBR).

     CMBR is the background radiation present throughout the cosmos. It maintains a Planckian spectrum at a temperature $2.7K$. This radiation comes from the epoch of decoupling, which took place around $t_{dec} \approx 3,80,000$ years.

     Due to expansion, the universe cooled down to a temperature below the ionization energy of the light elements like hydrogen and helium. At this condition, the electrons and protons were able to combine and form atoms. Once this happened, the photons became free of interactions, i.e. they decoupled. Since then, the photons have been traveling through the universe. Those are the photons we see as the CMBR in the sky. The CMBR is isotropic to 1 part in 10^5.

- **Hubble's law**

  Hubble's law gives the relation between the recession velocity and distance of distant cosmic structures. It can be stated as:

  The recession velocity of distant galaxies in the sky is directly proportional to their distance.

  Mathematically, if V is the recession velocity and D is the distance, than according to Hubble's law

  $$V \propto D\,.$$

  We can write

  $$V = H_0 D\,,$$

  where H_0 is the constant of proportionality known as the **Hubble constant**.

  This says the farther the galaxy is, the faster it recedes from the point of observation. It is applicable everywhere in the universe (except for the gravitationally bound structures.)

  The implication of Hubble's law is astonishing. It implies **our universe is expanding**, i.e., the distance between distant objects in our universe is increasing with time.

  Actually, the Hubble parameter is a function of time. We write the Hubble parameter as $H(t)$ and $H_0 = H(t_0)$, where t_0 is the present time.

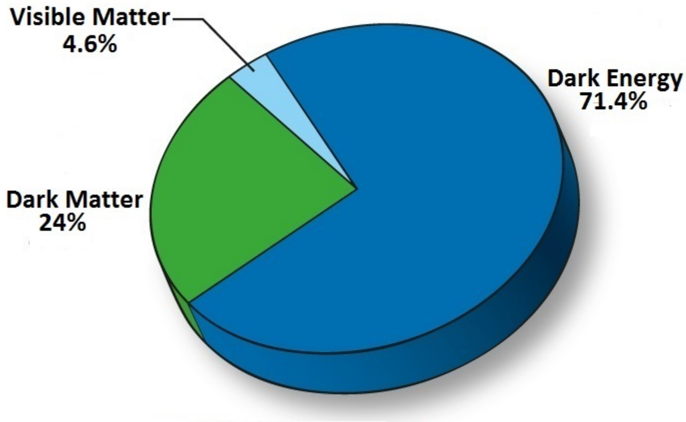
  The Hubble parameter $H(t)$ and the scale factor $R(t)$ are related as

  $$H(t) = \frac{\dot{R}(t)}{R(t)}\,.$$

- **Composition of the Universe**

  Cosmological observations shows that our universe is not just made up of visible matter and radiation but there is also the presence of dark matter and dark energy. The WMAP(Wilkinson Microwave Anisotropy Probe) survey and the supernova data gives the relative composition of visible matter, dark matter and dark energy in our universe. The WMAP data [3], as of January 2013, revels that our universe consists of

  - 4.6% **Visible Matter -** This is in the form of protons, nuclei and electrons.
  - 24% **Cold Dark Matter -** This is the invisible matter content. Dark matter primarily interacts by means of gravity. It does not interact with electromagnetic radiation, it is invisible, which is why the name dark. Its presence is only inferred by the gravitational effects as its other interactions are very weak.
  - 71.4% **Dark Energy -** This is the dominant form of energy in our universe today. Its properties are unknown. Since our universe is currently dark energy dominated, and its properties are not known, it prohibits any certain prediction about the future evolution of our universe.



Pie chart showing the content of our universe today

1.2 Einstein's Field Equation and FRW Cosmology

1.2.1 Einstein's Field Equation

Einstein's field equation in general relativity (GR) which determines the dynamics of spacetime in response to the presence of mass-energy density is given by (Ref. [20])

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{1.1}$$

which can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor for the spacetime, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is known as the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor.

In GR, the energy-momentum tensor for a perfect fluid can be represented as a covariant tensor as

$$T^{\mu\nu} = (P+\rho)u^{\mu}u^{\nu} - g^{\mu\nu}P\,. \tag{1.3}$$

In a co-moving reference frame, u^{μ}, the four-velocity is $u^{\mu} = (1,0,0,0)$ and

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \tag{1.4}$$

where ρ and P represent the energy density and pressure respectively.

1.2.2 RW Metric and Friedmann Solution

The Robertson-Walker (RW) metric for an isotropic and homogeneous spacetime can be represented in spherical coordinate (r,θ,ϕ) as

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.5}$$

where $R(t)$ is the scale factor and k is the spatial curvature parameter of space.

- For a non-static universe, the physical distance $d(t)$ between two point at time t can be written as

  $$d(t) = d(t_i)\frac{R(t_i)}{R(t)}\,,$$

  where t_i is some earlier time. The covariant metric tensor of the above RW metric is

  $$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -R^2(1-kr^2)^{-1} & 0 & 0 \\ 0 & 0 & -R^2r^2 & 0 \\ 0 & 0 & 0 & -R^2r^2\sin^2\theta \end{bmatrix}. \tag{1.6}$$

- **Friedmann Equations**

  Using the metric tensor (1.6) and the energy-momentum tensor (1.4) in the Einstein's field equation, (1.1) gives the following set of equations:

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{1.7}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = -8\pi G P\,. \tag{1.8}$$

  Eq. (1.7) and eq. (1.8) are known as the 1^{st} and 2^{nd} Friedmann equation respectively.

- **Energy-Momentum Conservation Equation**

  In an adiabatic system,

$$dQ = 0\,.$$

  Now $E = \rho V$ and $V \sim R^3$.

  From the 1^{st} law of thermodynamics,

$$dQ = 0 \quad \Rightarrow \quad dE + PdV = 0 \tag{1.9}$$

$$\Rightarrow \quad d\left(\rho R^3\right) + Pd\left(R^3\right) = 0 \tag{1.10}$$

$$\Rightarrow \quad 3R^2\dot{R}\rho + R^3\dot{\rho} + 3PR^2\dot{R} = 0 \tag{1.11}$$

$$\Rightarrow \quad \dot{\rho} + 3\frac{\dot{R}}{R}(\rho P) = 0\,. \tag{1.12}$$

  Eq. (1.7), eq. (1.8) and eq. (1.12) are related by the Bianchi identity. Generally we take eq. (1.7) and eq. (1.12) as fundamental equations.

- **Derivation of eq.**(1.8) **using eq.** (1.7) **and eq.** (1.12)

  Differentiating eq. (1.7) with respect to time gives

$$\frac{2\dot{R}\ddot{R}}{R^2} - 2\frac{\dot{R}^3}{R^3} - 2\frac{k\dot{R}}{R^3} = \frac{8\pi G}{3}\dot{\rho}\,. \tag{1.13}$$

  Replacing $\dot{\rho}$ by $-3\frac{\dot{R}}{R}(\rho + P)$ in (1.13) results in

$$\frac{\ddot{R}}{R} - \left[\frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right] = -4\pi G(\rho + P)\,. \tag{1.14}$$

  Using eq. (1.7) in eq.(1.14) gives

$$\frac{\ddot{R}}{R} = -4\pi G(\rho + P) = -\frac{1}{2}\left(\frac{8\pi G}{3}\rho\right) - 4\pi G p = -\frac{4\pi G}{3}(\rho + 3P)\,. \tag{1.15}$$

  Again using eq. (1.7) in eq. (1.15) results in

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = -8\pi G P\,, \tag{1.16}$$

  which is nothing but the 2^{nd} Friedmann equation.

- From eq. (1.15), we can see that if $P < -\frac{1}{3}\rho$, one gets an accelerated universe.

1.3 Non-Static Models of the Universe

In this section we will consider the time evolution of $R(t)$ for a matter dominated universe with spatial curvature parameter $k = -1, 0$, and 1.(Ref. [6])

In a matter dominated universe, the energy density is dominated by that of non-relativistic particles and pressure $P = 0$.

Friednmann equations for a matter dominated universe take the form

$$\frac{\dot{R}^2 + k}{R^2} = \frac{8\pi G}{3}\rho\,, \tag{1.17}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = 0\,. \tag{1.18}$$

Substituting eq. (1.17) in eq. (1.18) gives

$$2\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}\rho\,. \tag{1.19}$$

We can write eq. (1.17) as

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho\,, \tag{1.20}$$

or equivalently

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho\,, \tag{1.21}$$

or

$$\frac{k}{R^2} = \frac{8\pi G}{3}\rho - H^2 = \frac{8\pi G}{3c^2}\left[\rho - \frac{3H^2}{8\pi G}\right] = \frac{8\pi G}{3}\left[\rho - \rho_c\right]\,, \tag{1.22}$$

where ρ_c , the critical density, defined as

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1.23}$$

is the energy density for a flat universe.

For an isolated system of homogeneous non-relativistic particles, ρ, the energy density is is given by $\rho = mn$, where m is the mass of the particles and n is the number density of the particles. Since the volume varies as R^3, n varies as R^{-3}. This implies $\rho(t) \sim R^{-3}(t)$.

Another useful quantity, the deceleration parameter is defined as

$$q(t) = -\frac{\ddot{R}(t)R(t)}{\dot{R}^2(t)}\,. \tag{1.24}$$

Using eq. (1.19) and eq. (1.24), the decelerating parameter $q(t)$ can be written as

$$q(t) = -\frac{\ddot{R}(t)R(t)}{\frac{\dot{R}^2(t)}{R^2(t)}R^2(t)} = \frac{\ddot{R}(t)}{H^2(t)R(t)} = \frac{1}{H^2(t)}\frac{4\pi G\rho_0 R_0^3(t)}{3R^3(t)} = \frac{8\pi G\rho_0 R_0^3}{3H^2(t)R^3(t)}\,. \tag{1.25}$$

For the present time, we have

$$q_0 = \frac{8\pi G\rho_0 R_0^3}{6H_0^2(t)R_0^3(t)} = \frac{\rho_0}{2\rho_c}\,.$$

- **Case 1 - Closed Universe**

  For a closed Universe, we have $k = 1$, $\rho > \rho_c$, $q > \frac{1}{2}$.

  For $k = 1$, eq. (1.7) gives

$$\frac{\dot{R}^2 + 1}{R^2} = \frac{8\pi G \rho_0 R_0^3}{3R^3} \tag{1.26}$$

$$\dot{R}^2 + 1 = \frac{8\pi G \rho_0 R_0^3}{3R} \tag{1.27}$$

$$\dot{R}^2 + 1 = \frac{B}{R} \, , \tag{1.28}$$

  where we define

$$B = \frac{8\pi G \rho_0 R_0^3}{3R} \, . \tag{1.29}$$

  Now from eq. (1.28),

$$\dot{R}^2 + 1 = \frac{B}{R} \quad \Rightarrow \quad \frac{dR}{dt} = \sqrt{\frac{B - R}{R}} \tag{1.30}$$

$$\Rightarrow \quad \int_0^t dt = \int_0^R \sqrt{\frac{R}{B - R}} dR \, . \tag{1.31}$$

  Using an angular parameter η , we write

$$R = B \sin^2 \frac{\eta}{2} = \frac{B}{2} \left(1 - \cos\eta\right) \tag{1.32}$$

$$\Rightarrow dR = B \sin\frac{\eta}{2} \cos\frac{\eta}{2} d\eta \, . \tag{1.33}$$

  Eq. (1.31) on simplification gives

$$R = \frac{B}{2} \left(1 - \cos\eta\right) \, , \tag{1.34}$$

  and

$$t = \frac{B}{2} \left(\eta - \sin\eta\right) \, . \tag{1.35}$$

- **Case 2 - Flat Universe**

  For a flat universe we have $k = 0$, $\rho = \rho_c$, $q = \frac{1}{2}$.

  Now from eq. (1.7), with $k = 0$, we have

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G \rho_0 R_0^3}{3R^3} \, . \tag{1.36}$$

  From eq. (1.23), we have

$$H^2 = \frac{8\pi G \rho_0}{3} \, . \tag{1.37}$$

  So eq. (1.36) becomes

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G \rho_0 R_0^3}{3R^3} = \frac{H^2 R_0^3}{R^3} \tag{1.38}$$

or

$$R\dot{R}^2 = H^2 R_0^3 \Rightarrow \sqrt{R}\dot{R} = \left(H^2 R_0^3\right)^{\frac{1}{2}} \,. \tag{1.39}$$

Integrating eq. (1.39) we get

$$R(t) = \left[\frac{3}{2}\left(H^2 R_0^3\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} t^{\frac{2}{3}} \,. \tag{1.40}$$

This gives

$$t = \frac{2}{3\left(H^2 R_0^3\right)^{\frac{1}{2}}} R^{\frac{3}{2}}(t) \,. \tag{1.41}$$

So we have

$$t_0 = \frac{2}{3H_0} \,. \tag{1.42}$$

For a universe dominated by relativistic particles, i.e., a radiation dominated universe,

$$R(t) \sim t^{\frac{1}{2}}$$

and

$$t_0 = \frac{1}{2H_0} \,.$$

- **Case 3 - Open Universe**

For an open universe we have $k = -1\,,\ \rho < \rho_c\,,\ q < \frac{1}{2}$.

For $k = -1$, eq. (1.7) gives

$$\dot{R}^2 - 1 = \frac{8\pi G \rho_0 R_0^3}{3R} = \frac{B}{R} \tag{1.43}$$

$$\Rightarrow \frac{dR}{dt} = \sqrt{\frac{B+R}{R}} \tag{1.44}$$

$$\Rightarrow \int_0^t dt = \int_0^R \sqrt{\frac{R}{B+R}}\, dR \,. \tag{1.45}$$

Introducing an angular parameter η, we write

$$R = B\sinh^2\frac{\eta}{2} = \frac{B}{2}\left(1 - \cosh\eta\right) \tag{1.46}$$

$$\Rightarrow dR = B\sinh\frac{\eta}{2}\cosh\frac{\eta}{2} d\eta \,. \tag{1.47}$$

Eq. (1.45) on simplification gives

$$R = \frac{B}{2}\left(\cosh\eta - 1\right) \,, \tag{1.48}$$

and

$$t = \frac{B}{2}\left(\sinh\eta - \eta\right) \,. \tag{1.49}$$

Chapter 2

Inflation

2.1 Problems with the Standard Cosmology

The standard model of cosmology is able to explain many of the observational facts about our universe with great success. For instance, it successfully explains the followings.(Ref. [4, 5, 10–13, 18, 19])

- The expansion of the universe.
- The age of the universe.
- The origin and spectrum of the Cosmic Microwave. Background Radiation (CMBR).
- The origin and abundance of light elements in the universe.

However, the standard theory of cosmology leaves some of the crucial observational facts about the universe unexplained, such as the Horizon and Flatness problem.

2.1.1 The Horizon Problem

Apparently causally disconnected regions in the universe have the same temperature.

The observation of CMBR revels that apparently disconnected regions in the sky have the same background temperature. As mentioned earlier, the background temperature is the radiation from the epoch of decoupling, which took place around $t_{dec} = 3,80,000$ year. The horizon size of the universe at t_{dec} subtends an angle of 1° in the sky today. So we expect that only the regions of sky lying within 1° will have the same temperature because such regions were in causal contact and hence in thermal equilibrium at t_{dec}. But observations indicate that the CMBR is isotropic. Now how can two regions in the sky separated by greater than 1°, which were presumably not in causal contact at decoupling, have the same temperature? The standard theory of cosmology does not answer this.

2.1.2 The Flatness Problem

From the 1st Friedmann equation,

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \Rightarrow H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \Rightarrow 1 + \frac{k}{H^2R^2} = \frac{8\pi G}{3H^2}\rho\,, \tag{2.1}$$

or

$$\frac{k}{H^2R^2} = \frac{\rho}{\rho_c} - 1 = \Omega - 1\,, \tag{2.2}$$

where

$$\Omega = \frac{\rho}{\rho_c}$$

is the energy density parameter, defined as the ratio of energy density and critical energy density.

Cosmological observations show our universe is nearly flat , i.e $\rho \approx \rho_c$. Then Ω, at t_{pl} would have to be very close to 1. This seems like the initial conditions of the universe were finely tuned, which is why the problem is also known as the ***fine tuning problem***.

2.2 Introduction to the Theory of Inflation

The theory of inflation tries to provide a solution to the above problems. It postulates the existence of a scalar field ϕ, the **inflaton**, with which was associated a potential $V(\phi)$. The energy density associated with ϕ, represented as $\rho(\phi)$ was the dominant form of energy density at some early time. Since the evolution of the universe at any time is determined by the dominant energy density, $\rho(\phi)$ determined the evolution of the universe as long as it remained dominant.

We can write the energy density of ϕ as the contribution of both the kinetic energy density ρ_K and potential energy density ρ_P, i.e.,

$$\rho(\phi) = \rho_K + \rho_P\,. \tag{2.3}$$

Initially, for some reason, the scalar field was not at the minimum of its potential. It was displaced. Then, because its potential was flat, it rolled slowly to the minimum of its potential, the state of its lowest energy.

During the slow roll period, we can neglect the contribution of ρ_k and write

$$\rho(\phi) \approx \rho_P\,. \tag{2.4}$$

Inflation says, for a short period of time, when ϕ was slowly rolling down, $\rho(\phi)$ was nearly constant. Now from the 1st Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_\phi\,, \tag{2.5}$$

Taking the RHS as a constant quantity, the above equation can be integrated to give

$$R(t) = R(t_i)e^{A(t-t_i)}\,, \tag{2.6}$$

where t_i is the initial time of inflation, and $A^2 = \frac{8\pi G}{3}\rho_\phi$.

This short period of exponential increase in the value of scale factor R(t), is known as the period of **inflation**.

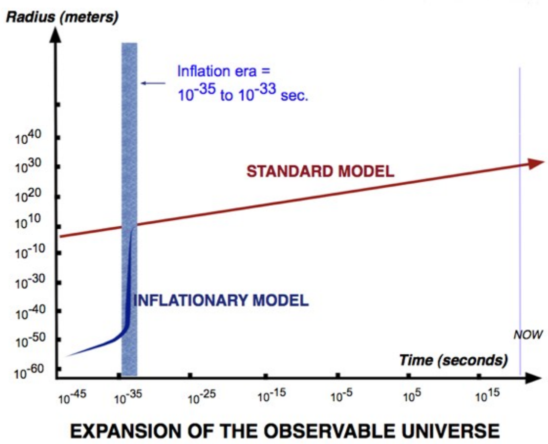


Figure 2.1: The shape of inflaton potential in large field models. The pink circle denotes the evolution of ϕ. (Ref [2])

2.3 Solution to the Problems with Standard Cosmology

The exponential increase in the scale factor easily solves the horizon and flatness problem associated with the standard cosmology. (Ref. [4, 8, 9, 11, 18])

- **Solution to the Horizon problem**

  Because of the exponential expansion of the universe, the causal horizon size increased by a huge factor. The Hubble radius H^{-1} at t_{dec} was smaller than the causal horizon at t_{dec}.

  The apparently causally disconnected regions at t_{dec} have the same background temperature today because they were in fact part of a region that was in causal contact at the beginning of the inflation.

- **Solution to the Flatness problem**

  The exponential expansion smoothed out the curvature.

  From eq. (2.2),

  $$\frac{k}{H^2 R^2} = \Omega - 1\,.$$

  It is easy to see that due to the exponential increase of the scale factor, whatever the value of curvature of space $\left(= 6\frac{k}{R^2}\right)$ was before inflation, inflation reduced it to almost zero. In other words, at the end of inflation Ω was very close to 1, and it continues to be so.

Chapter 3

Inflaton: Dynamics and Relevant Parameters

As has been first proposed by Guth [9] and Linde [17], The theory of inflation solves the **Horizon**, **Flatness** and **Monopole** problem associated with the FRW cosmology. Additionally provides the seed for the **Large Scale Structure (LSS)** formation in the universe.

Inflation postulates the existence of a scalar field ϕ, the inflaton. Associated with inflaton was the potential $V(\phi)$. The essential condition for the inflation to occur is the **"Slow Roll"**. To extract some observable /detectable evidence, one needs a particular form of inflaton. Unfortunately, inflation says nothing about the form of $V(\phi)$. The potential $V(\phi)$ can be any suitable potential. This is the reason today there exist hundreds of models for inflation, each with a different form of $V(\phi)$. The good thing is, they have different predictions which, when analyzed with the experimental/observational data, helps us the discard/accept any model.

As we have learned [12], the essential condition for the inflation to occur is the slow roll of $V(\phi)$. Now onwards We will denote $V(\phi)$ as V. Mathematically, the **S**low **R**oll **A**pproximations (**SRA**) are

1. V was nearly constant.
2. $\dot{V}$ and $\ddot{V}$ can be taken to be 0 or can be safely neglected in comparison to V.
3. We can take the gradient of the potential $V' = \frac{V}{\phi}$.

We have the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{3.1}$$

The energy-momentum tensor from Noether's theorem can be written as

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}\,, \tag{3.2}$$

where the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)\,. \tag{3.3}$$

For a perfect fluid

$$T^{\mu\nu} = diag(\rho, p, p, p)\,. \tag{3.4}$$

This gives

$$\rho = T^{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{3.5}$$

and

$$p = \frac{T^{11} + T^{22} + T^{33}}{3} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.6}$$

Invoking the **SRA**, we can easily verify $\rho = -p$.

Using eq. (3.5) and eq. (3.6) in eq. (3.1), we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) + 3H\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right) &= 0 \\ \ddot{\phi}\dot{\phi} + V'\dot{\phi} + 3H\dot{\phi}^2 &= 0 \\ \ddot{\phi} + 3H\dot{\phi} &= -V'. \end{aligned} \tag{3.7}$$

Invoking the **SRA** we can write, eq. (3.7) as

$$3H\dot{\phi} = -V' \tag{3.8}$$

3.1 Slow Roll Parameters

The slow roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ are defined as

$$\epsilon(\phi) = \frac{M_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2\,, \tag{3.9}$$

$$\eta(\phi) = \frac{M_{pl}^2}{8\pi}\left(\frac{V''}{V}\right)\,. \tag{3.10}$$

Inflation will occur as long as ϵ and η are < 1 .

Now, Friedmann equation can be written as

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(V(\phi) + \frac{1}{2}\dot{\phi}^2\right) \tag{3.11}$$

Using **SRA**, we write,

$$H^2 = \frac{8\pi G}{3}V(\phi) \tag{3.12}$$

3.2 Slow roll parameter and Inflation

In this section we will derive the relation between the slow roll parameter and inflation.

The Hubble parameter H is defined as

$$H = \frac{\dot{R}}{R}\,. \tag{3.13}$$

This gives,

$$\dot{H} = \frac{\ddot{R}}{R^2} - \left(\frac{\dot{R}}{R}\right)^2\,. \tag{3.14}$$

Since, $\ddot{R}$ & $R > 0$, we get $\dot{H} + H^2 > 0$, i.e., and $-\frac{\dot{H}}{H^2} < 1$.

Substituting G by $\frac{1}{M_{pl}^2}$ in eq. (3.12), we get

$$H^2 = \frac{8\pi V(\phi)}{3M_{pl}^2} \,. \tag{3.15}$$

On differentiation. eq.(3.15) gives,

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi}{3M_{pl}^2}\frac{d}{dt}V \\ &= \frac{8\pi}{3M_{pl}^2}\frac{dV}{d\phi}\frac{d\phi}{dt} \\ &= \frac{8\pi\dot{\phi}V'}{3M_{pl}^2} \,. \end{aligned} \tag{3.16}$$

So we have

$$\dot{H} = \frac{8\pi\dot{\phi}V6\prime}{6HM_{pl}^2} \,. \tag{3.17}$$

We have seen that, for inflation to occur we need

$$-\frac{\dot{H}}{H^2} < 1 \,.$$

Using eq. (3.15) and (3.17) we get

$$-\frac{8\pi\dot{\phi}V6\prime}{6HM_{pl}^2H^2} < 1$$

$$\Rightarrow -\frac{\dot{\phi}V'}{2HV} < 1 \,.$$

Also, we get

$$-\frac{\dot{H}}{H^2} = -\frac{\dot{\phi}V'}{2HV} = \frac{V'^2}{6H^2V} < 1 \,.$$

Using eq. (3.15) in the above expression gives,

$$\frac{V'^2}{6V\left(\frac{8\pi}{3M_{pl}^2}\right)V} < 1$$

$$\Rightarrow \frac{M_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 < 1 \,,$$

which is essentially the 1^{st} slow roll parameter. Hence we see that inflation will take place until ϵ reaches 1.

3.3 Number of e-fold increase

During inflation the scale factor of the universe changes as

$$R(t) = R(t_0)e^{Ht}.$$

The factor Ht is the number of e-fold increase in the scale factor. So we have, the no. of e-fold increase

$$N = Ht = H\int_{t_i}^{t_f} dt = ln\left(\frac{R(t_f)}{R(t_i)}\right) \tag{3.18}$$

Now, dividing eq. (3.7) by eq. (3.12) gives

$$\frac{3\dot{\phi}}{H} = -\frac{3V'M_{pl}^2}{8\pi} \Rightarrow \frac{\dot{\phi}}{H} = -\frac{M_{pl}^2}{8\pi}\left(\frac{V'}{V}\right). \tag{3.19}$$

So we have

$$N = \int_{t_i}^{t_f} H dt = \int_{t_i}^{t_f} H\frac{dt}{d\phi}d\phi = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}}d\phi\,. \tag{3.20}$$

From, eq. (3.19) and eq. (3.20),

$$N = \int_{\phi_i}^{\phi_f} -\frac{8\pi}{M_{pl}^2}\left(\frac{V}{V'}\right)d\phi = \int_{\phi_f}^{\phi_i} \frac{8\pi}{M_{pl}^2}\left(\frac{V}{V'}\right)d\phi\,, \tag{3.21}$$

gives the number of e-fold increase in the size of the universe during the period of inflation in terms of the potential and its derivative

In the next chapter we discuss several important scalar field models in details and check if they can be a potential inflaton.

Chapter 4

Higgs as a Potential Inflaton

The recent discovery of Higgs [1] tempts us to think that Higgs might be the scalar field responsible for the inflation. In what follows, we will explore this possibility. We will now examine if Higgs field can be a possible candidate for Inflaton or not (Ref. [4]).

The Higgs potential can be written as

$$V = \lambda(\phi^2 - v^2) \tag{4.1}$$

This gives

$$V' = 4\lambda\phi(\phi^2 - v^2) \tag{4.2}$$

and

$$V'' = 4\lambda(3\phi^2 - v^2)\,. \tag{4.3}$$

Using the definitions of slow roll parameters from (3.9) and (3.10), we get

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 = 8M_{pl}^2\frac{\phi^2}{(\phi^2 - v^2)^2}\,, \tag{4.4}$$

and

$$\eta = 4M_{pl}^2\frac{3\phi^2 - v^2}{(\phi^2 - v^2)^2}\,. \tag{4.5}$$

For inflation to take place, the parameters should be less than 1.

4.1 Near the Hilltop

Between $\phi = 0$ to $\phi = v$, the potential looks flat. However, for $\phi << v$, the slow roll parameters takes the value,

$$\epsilon = 8M_{pl}^2\frac{\phi^2}{(\phi^2 - v^2)^2} \approx 8M_{pl}^2\frac{\phi^2}{v^4} \approx \left(\frac{M_{pl}\phi}{v}\right)^2 \tag{4.6}$$

$$\eta = 4M_{pl}^2\frac{3\phi^2 - v^2}{(\phi^2 - v^2)^2} \approx \left(\frac{M_{pl}}{v}\right)^2\,. \tag{4.7}$$

Taking the numerical values $M_{pl} = 2.4 \times 10^{18}$ GeV and $v = 246$ GeV, we find that no where in between $\phi = 0$ and $\phi = v$, both ϵ and η becomes small.

4.2 Large-Field Region

For large field, i.e. $\phi^2 >> v$, the potential looks like $V = \lambda\phi^4$ and we get

$$\epsilon = \frac{8M_{pl}^2}{\phi^2} \tag{4.8}$$

and

$$\eta = \frac{12M_{pl}^2}{\phi^2} \tag{4.9}$$

Here we get a slow roll condition when, $\phi > 8M_{pl}$.

The end of inflation is defined to be when ϵ reaches 1. This condition gives

$$\phi_E = 2\sqrt{2}M_{pl}\,. \tag{4.10}$$

Now no. of e-folds is calculated as

$$N = \int_{\phi_i}^{\phi_E} \frac{H}{\dot{\phi}} d\phi = -\int_{\phi_i}^{\phi_E} \frac{1}{\sqrt{2\epsilon}} \frac{d\phi}{M_{pl}} \approx \frac{\phi^2}{8M_{pl}^2}\,. \tag{4.11}$$

We need N=60, so we have $\phi_i = 22M_{pl}$

The amplitude of scalar fluctuation is given by,

$$\Delta_s^2 = \frac{1}{8\pi^2}\frac{1}{\epsilon}\frac{H^2}{M_{pl}^2} \tag{4.12}$$

Using the eq. (3.12) and (4.8), we have

$$\Delta_s^2 = \frac{1}{8\pi^2}\frac{1}{\epsilon}\frac{H^2}{M_{pl}^2} = \frac{1}{8\pi^2}\frac{\phi^2}{8M_{pl}^2}\frac{V}{3M_{pl}^4} = \frac{\lambda\phi^6}{8\pi^2 \times 24M_{pl}^6} = \frac{1}{8\pi^2}\frac{\lambda(8M_{pl}^2N)^3}{24M_{pl}^6} = \frac{8\lambda N^3}{3\pi^2} \tag{4.13}$$

The Higgs mass is measured around $\phi = v$. We have

$$m_H^2 = V''(\phi = v) = 8\lambda v^2\,. \tag{4.14}$$

So we have the scalar fluctuation,

$$\Delta_s^2 = \frac{N^3}{3\pi^2}\left(\frac{m_H}{v}\right)^2 \tag{4.15}$$

Numerical requirements are $N = 60$ and $v = 246$. This gives,

$$\Delta_s^2 = 0.12\left(\frac{m_H}{GeV}\right)^2\,. \tag{4.16}$$

Δ_s^2 is measured to be 2×10^{-9}. This gives $m_H \approx 10^{-4}$,BUT from experiment, $m_H = 125$Gev [1].

So Higgs may not (?) possibly be the inflaton.

However, there is a possibility that Higgs can still drive the inflaton by coupled with gravity. In the next chapter we discuss this possibility.

Chapter 5

Higgs Coupled with Gravity

As we have seen before, the Higgs potential

$$V = \lambda(\phi^2 - v^2) \tag{5.1}$$

(itself) can not drive the inflation. We now test if Higgs non-minimally coupled with gravity can be a potential inflaton (Ref [4].)

The action for the non-minimal coupling of Higgs with gravity can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}f(\phi)R + \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right], \tag{5.2}$$

where we have

$$f(\phi) = 1 + \xi\frac{\phi^2}{M_{Pl}^2}.$$

The coupling brings few changes in the action. Defining $\tilde{g} = f(\phi)g$, We notice that for $\xi >> 1$, the action modifies to

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{Pl}^2}{2}\tilde{R} + \frac{1}{2}(\partial\Phi)^2 - V(\Phi)\right], \tag{5.3}$$

where we introduce

$$V(\Phi) \approx \frac{\lambda M_{Pl}^2}{4\xi^2}\left(1 - 2exp\left[-\sqrt{\frac{2}{3}}\frac{\Phi}{M_{Pl}}\right]\right) \tag{5.4}$$

with

$$\frac{\Phi}{M_{Pl}} = \sqrt{\frac{3}{2}}\,ln(f(\phi)).$$

5.1 Slow Roll Analysis

Now we perform the slow roll analysis of the modified potential in the region where $\Phi >> M_{Pl}$.

The first and second derivative of (5.4) gives

$$V'(\Phi) = \frac{\lambda M_{Pl}^2}{2\xi^2}\left[\sqrt{\frac{2}{3}}\,exp\left(-\sqrt{\frac{2}{3}}\frac{\Phi}{M_{Pl}}\right)\right] \tag{5.5}$$

and

$$V''(\Phi) = \frac{-\lambda M_{Pl}^2}{3\xi^2}\, exp\left(-\sqrt{\frac{2}{3}}\frac{\Phi}{M_{Pl}}\right)\,, \tag{5.6}$$

respectively.

From the definition, the slow roll parameters are

$$\epsilon_\nu = \frac{1}{2}M_{Pl}^2\left(\frac{V'}{V}\right)^2 \;\&\; \eta_\nu = M_{Pl}^2\left(\frac{V''}{V}\right). \tag{5.7}$$

Using (5.5) and (5.6) in (5.7), we compute

$$\epsilon_\nu = \frac{4}{3}\, exp\left(-2\sqrt{\frac{2}{3}}\frac{\Phi}{M_{Pl}}\right), \; \eta_\nu = -\frac{4}{3}\, exp\left(-\sqrt{\frac{2}{3}}\frac{\Phi}{M_{Pl}}\right). \tag{5.8}$$

In calculating the slow roll parameters, we have used the condition that $\Phi >> M_{Pl}$, in this regime the potential is nearly flat, and we are interested in this region.

From (5.8) we note that

$$\epsilon_\nu = \frac{3}{4}\eta_\nu^2. \tag{5.9}$$

Now we will calculate the number of e-folds during the inflation. For the region $\phi_\star >> \phi_E$, the number of e-folds, $N_\star$ is given by

$$\begin{aligned} N_\star &= -\int_{\phi_\star}^{\phi_E} \frac{d\phi}{M_{Pl}\sqrt{2\epsilon_\nu}} \\ &= \int_{\phi_E}^{\phi_\star} \frac{d\phi}{M_{Pl}}\sqrt{\frac{3}{8}}\, exp\left(\sqrt{\frac{2}{2}}M_{Pl}\right) \\ &= \frac{3}{4}\, exp\left(\sqrt{\frac{2}{3}}\frac{\Phi_\star}{M_{Pl}}\right) \\ &= -\eta^{-1}. \end{aligned} \tag{5.10}$$

The spectral index is given by

$$n_s = 1 - 6\epsilon_v + 2\eta_v, \tag{5.11}$$

which using (5.9), can be written as

$$n_s = 1 + 2\eta_v. \tag{5.12}$$

Using the relation $N_\star = -\eta^{-1}$ we can write

$$n_s = 1 - \frac{2}{N_\star}.$$

The amplitude of the power spectrum is given by

$$\Delta_s^2 = \frac{1}{8\pi^2}\frac{1}{\epsilon_v}\frac{H^2}{M_{Pl}^2}. \tag{5.13}$$

Using (5.9), we calculate

$$\begin{aligned}
\Delta_s^2 &= \frac{1}{8\pi^2}\frac{1}{\epsilon_v}\frac{H^2}{M_{Pl}^2} \\
&= \frac{1}{8\pi^2}\frac{4}{3}\eta_v^{-2}\frac{V}{3M_{Pl}^4} \\
&\approx \frac{1}{8\pi^2}\frac{4}{3}N_\star^2\frac{\lambda}{12\xi^2} \\
&= \frac{N_\star^2\lambda}{72\pi^2}\frac{1}{\xi^2} \\
&= 2\times 10^{-9}. \qquad (5.14)
\end{aligned}$$

We need $N_\star \sim 60$, this implies, $\xi \sim 5\times 10^4\sqrt{\lambda}$.

The gravitational coupling should not affect Higgs at the energy scale the LHC operates. We have the relation

$$m_H^2 = 8\lambda v^2,$$

and we know that the Higgs mass is measured to be approximately $m_H = 125\text{GeV}$, and the value of V is 246 GeV.

So we have

$$\lambda = \frac{m_H^2}{8v^2} = \frac{125^2}{8\times 246^2} = 0.03223.$$

This gives

$$\lambda = 50000\times\sqrt{0.03223} = 8976.3578 \approx 9\times 10^3.$$

We find that the coupling constant between the Higgs and gravity is quite large, which seems unnatural, the structure of the theory may be destabilized by the quantum corrections at energy scale lower than the inflationary scale.

In essence, though the Higgs coupling with Gravity seems a possible model of inflaton, the high value of coupling constant makes it not so much better than other model of inflaton.

5.2 Few Other Important Models of Inflaton

There exist numerous models of inflaton. Below we shed some light on few of them.

1. **Small field inflaton**
   The small field models postulate that field moves over a small distance $\Delta\phi < M_{Pl}$. This predicts the gravitational waves produced by the inflation is quite small to detect. This kind of model often arises mainly due to spontaneous symmetry breaking. Examples are:

   (a) Higgs like potential

   $$V(\phi) = V_0\left[1-\left(\frac{\phi}{\mu}\right)^2\right]^2$$

(b) The Coleman-Weinberg potential

$$V(\phi) = V_0 \left[\left(\frac{\phi}{\mu} \right)^4 \left(ln \left(\frac{\phi}{\mu} \right) - \frac{1}{4} \right) + \frac{1}{4} \right].$$

2. **Large field inflation**
   In this kind of models, the inflation starts at large values and then evolves to a minimum at $\phi = 0$. If the evolution starts at a value of ϕ such that $\Delta\phi(\phi_f - \phi_i) > M_{Pl}$, the gravitational waves produced by the inflation should be detectable.

   The normal class of inflatons in the large field models look like

   $$V(\phi) = \lambda_p \phi^p.$$

   The slow roll parameters in this type of potential are small for super Planckian region $\phi >> M_{Pl}$. An example of large field inflaton is the Natural Inflaton which is given by

   $$V(\phi) = V_0 \left[\cos \left(\frac{\phi}{f} \right) + 1 \right].$$

   This kind of potential arises due to the axion field. Note that depending on the value of f, the potential $V(\phi)$ can be small field or large field.

3. **Non-minimal coupling with gravity**
   In this kind of models the inflation field is coupled with gravity by a coupling constant. We have discussed the Higgs coupling with gravity in the last section (5) .

4. **Modified Gravity Models**
   There is a possibility that the standard Einstein-Hilbert action in general relativity modifies at higher energy levels. One such example is the $f(R)$ gravity theory, where the Ricci scalar R in the standard Einstein-Hilbert action

   $$S = \frac{1}{k} \int R\sqrt{-g} d^4x$$

   is replaced by the a function of R, i.e. the new action takes the form

   $$S^\star = \frac{1}{2k} \int f(R)\sqrt{-g} d^4x.$$

5. **Non-minimal kinetic term**
   The action may contain non-canonical kinetic terms,

   $$\mathcal{L}_\phi^\star = F(\Phi, X) - V(\Phi)$$

   instead of $\mathcal{L}_\phi = X - V(\phi)$. In this kind of models the inflation can be driven by the kinetic term even in presence of step potential.

6. **Multiple field inflaton**
   This kind of models based on the assumption that the inflaton potential may contain more than one type of fields. We discuss one such inflaton in later section.

   In the following chapters we study few more models in detail.

Chapter 6

Other Important Models of Inflaton

6.1 Large Field Inflation: Chaotic Inflation Model

The general type of large field potential are of type (Ref. [7])

$$V = g\phi^n,\, n > 0\,. \tag{6.1}$$

The coupling constant g has the dimension $[g] = (mass)^{4-n}$.

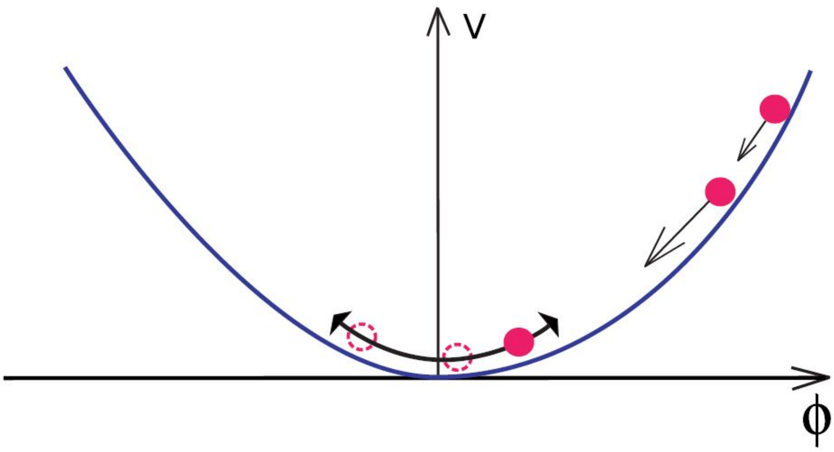


Figure 6.1: The shape of inflaton potential in large field models. The pink circle denotes the evolution of ϕ.

Now,

$$V' = gn\phi^{n-1},\, V'' = gn(n-1)\phi^{n-2}. \tag{6.2}$$

This gives the slow roll parameter

$$\begin{aligned}\epsilon &= \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \\ &= \frac{M_{Pl}^2}{16\pi}\left(\frac{gn\phi^{n-1}}{g\phi^n}\right)^2 \\ &= \frac{M_{Pl}^2}{16\pi}\left(\frac{n}{\phi}\right)^2 \end{aligned} \tag{6.3}$$

We need $\epsilon << 1$, for inflation, so

$$\begin{aligned}\epsilon << 1 &\Rightarrow \frac{M_{Pl}^2}{16\pi}\left(\frac{n}{\phi}\right)^2 << 1 \\ &\Rightarrow \frac{M_{Pl}}{4\sqrt{\pi}}\frac{n}{\phi} << 1 \\ &\Rightarrow \phi >> \frac{M_{Pl}n}{4\sqrt{\pi}}. \end{aligned} \tag{6.4}$$

In essence, it says that the inflation takes place if the field takes super-Planckian value

$$\frac{M_{Pl}n}{4\sqrt{\pi}}.$$

Fortunately, this does not requires the as quantum dynamics as the quantum gravity effects are not so effective when the field value $V(\phi)$ is sub-Planckian;

$$\begin{aligned}V(\phi) << M_{Pl}^4 &\Rightarrow g\phi^n << M_{Pl}^4 \\ &\Rightarrow \phi^n << G^{-1}M_{Pl}^4 \\ &\Rightarrow \phi << g^{-1/n}M_{Pl}^4 = \left(\frac{M_{Pl}^4}{g}\right)^{1/n}. \end{aligned} \tag{6.5}$$

From (6.4) and (6.5), we find that

$$\frac{nM_{Pl}}{4\sqrt{\pi}} << \phi << \left(\frac{M_{Pl}^4}{g}\right)^{1/n} \tag{6.6}$$

is the only condition satisfying simultaneously.

One possible picture of this large field inflation was first suggested by **Linde**. The idea can be simply put as follow:

The universe prior to inflation could have been highly inhomogeneous(scales exceeding l_{Pl}), it could have a strong spatial curvature(radius was of order l_{Pl}), high energy density, i.e. $\rho \sin M_{pl}^4$. The inflaton field assumed to obey the slow roll conditions at large value of ϕ). The effect of the curvature term in the Friedmann equation, although quite important, it's effect goes as the inverse of the scale factor, i.e. the curvature k varies as $k \sin 1/a^2$. As the universe expands, the scale factor increases, the curvature term decreases, but the scalar potential of ϕ remains the same. There may exist(or created due to quantum fluctuations) a small patch of space of size larger than Planckian size, whose spatial curvature terms in the Friedmann equation is small. This patch when starts to expand, soon enters the inflation regime. This happens in the Planckian epoch. The initial condition in this case here created accidentally, so it is named as *Chaotic inflation.*

Inflation lives as long as the slow roll conditions are satisfied. Once the inflation ends, the field ϕ starts to decay. The equation that governs the evolution of ϕ is

$$\ddot{\phi} + 3H\dot{\phi} + V(\phi) = 0. \tag{6.7}$$

During the slow roll, the Hubble friction term $3H\dot{\phi}$ is quite large. Once the Hubble friction term becomes small(compared to the $\ddot{\phi}$ term), we can safely neglect the friction term and this gives rise to a harmonic oscillator type equation;

$$\ddot{\phi} + V(\phi) = 0. \tag{6.8}$$

The field oscillates about the minimum of the potential and decays in this method. This decay of ϕ produces the particle in the standard model. This phase is called *reheating*. It may take several Hubble times.

The particles created out of the decay of ϕ contains high energy density. This creates a lot of heat and energy. This account for the large amount of entropy we have in the universe today.

We will now estimate the total duration of inflation and the number of e-folds occurred during the inflation. This can be done as follow.

The number of e-folds is given by

$$N_e(\phi) = \log\left[\frac{a_e}{a_\phi}\right] = \int_{t_\phi}^{t_e} H(t)dt = \frac{8\pi}{M_{Pl}^2}\int_{\phi_e}^{\phi}\frac{V}{V'}d\phi\,. \tag{6.9}$$

For power-law inflation potential,

$$V = g\phi^n,\ \frac{V}{V'} = \frac{g\phi^n}{gn\phi^{n-1}} = \frac{\phi}{n}.$$

The number of e-folds is than

$$N_e(\phi) = \frac{8\pi}{M_{Pl}^2}\int_{\phi_e}^{\phi}\frac{\phi}{n}d\phi = \frac{4\pi}{n}\frac{\phi^2}{M_{Pl}^2}. \tag{6.10}$$

So,

$$\phi^2 = \frac{N_e n M_{Pl}^2}{4\pi} \Rightarrow \phi(N_e) = \sqrt{\frac{nN_e}{4\pi}}M_{Pl}. \tag{6.11}$$

This essentially gives,

$$\phi(N_e) = \begin{cases} = 2.8 - 3.1\,M_{Pl} & \text{for} \quad n = 2, N_e = 50 - 60. \\ = 4.0 - 4.4\,M_{Pl} & \text{for} \quad n = 4, N_e = 50 - 60. \end{cases}$$

6.1.1 Relation between the slow roll parameter and number of e-folds

We have the slow roll parameter

$$\epsilon = \frac{M_{Pl}^2}{16\pi}\left(\frac{n}{\phi}\right)^2$$

and

$$\eta = \frac{M_{Pl}^2}{8\pi}\frac{V''}{V} = \frac{M_{Pl}^2}{8\pi}\frac{gn(n-1)\phi^{n-1}}{g\phi^n} = \frac{M_{Pl}^2}{8\pi}\frac{n(n-1)}{\phi^2}.$$

With

$$N_e(\phi) = \frac{4\pi}{n}\frac{\phi^2}{M_{Pl}^2}$$

we can write

$$\epsilon = \frac{\eta}{4N_e} \,\&\, \eta = \frac{n-1}{2N_e}.$$

Assume that the inflation begins when the energy density of the inflaton was comparable to Planck value, i.e. $V(\phi) = M_{Pl}^4$.

For the power-law potential $V(\phi) = g\phi^n$

$$\phi \sim g^{-1/n} M_{Pl}^{4/n} \tag{6.12}$$

and

$$N_e(\phi)(total) = \frac{4\pi}{n}\left(\frac{M_{Pl}^4}{g}\right)^{2/n} \times \frac{1}{M_{Pl}^2} = \frac{4\pi}{n}\left(\frac{M_{Pl}^{4-n}}{g}\right)^{2/n}, \tag{6.13}$$

which says for small value of g, the value of N_e is large.

In particular, for quadratic potential $V = \frac{m^2\phi^2}{2}$,

$$N_e = \frac{4\pi}{2}\left(\frac{M_{Pl}^{4-2}}{m^{2/2}}\right)^{2/2} = 4\pi\frac{M_{Pl}^2}{m^2}. \tag{6.14}$$

And for quartic potential $V = \frac{\lambda}{4}\phi^4$,

$$N_e(total) = \frac{4\pi}{4}\left(\frac{M_{Pl}^{4-4}}{\lambda/4}\right)^{2/4} = \frac{4\pi}{4}\left(\frac{4}{\lambda}\right)^{1/2} = \frac{2\pi}{\sqrt{\lambda}}. \tag{6.15}$$

From observation, the measure density perturbation $\partial\rho/\rho$ obtained value is

$$\partial\rho/\rho = 5\times 10^{-5}\begin{cases} n=2 & \text{with} \quad m \sim 10^{-6} M_{Pl} \\ n=4 & \text{with} \quad \lambda \sim 10^{-13} \end{cases}$$

This says the number of e-folds increase during inflation

for quadratic potential,

$$N_e = 4\pi\frac{M_{Pl}^2}{10^{-12}M_{Pl}^2} \sim 10^{13},$$

and for quartic potential,

$$N_e = \frac{2\pi}{\sqrt{\lambda}} = 2\pi \times \sqrt{10^{13}} \sim 10^7.$$

6.1.2 The duration of inflation.

To calculate the duration of inflation, we proceed as follow.

$$\Delta t(total) = \int_{t_i}^{t_f} dt = \int_{\phi_i}^{\phi_f} \frac{d\phi}{\dot\phi} = \int_{\phi_i}^{\phi_f} \frac{3H}{V'(\phi)} d\phi = \int_{\phi_i}^{\phi_f} \left(\frac{8\pi V}{3}\right)^{1/2} \frac{1}{V'}\frac{d\phi}{\phi} = 3\sqrt{\frac{8\pi}{3}}\frac{1}{M_{Pl}}\int_{\phi_i}^{\phi_f}\frac{\sqrt{V}}{V'}d\phi \tag{6.16}$$

For quadratic potential, $n = 2$, $m \sim 10^{-6} M_{Pl}$, so

$$\Delta t \sim \frac{1}{M_{Pl}}\frac{\phi_i}{m} \sim \frac{M_{Pl}}{m^2} \sim 10^{-31\cdot} sec.$$

For quartic potential $n = 4$, $\lambda = 10^{-13}$, so

$$\Delta t \sim \frac{1}{M_{Pl}} \frac{1}{\sqrt{\lambda}} \log \frac{\phi_i}{\phi_f} \sim \frac{1}{M_{Pl}} \frac{1}{\sqrt{\lambda}} \log \frac{1}{\lambda} \sim 10^{-35} sec.$$

This is astonishing! A microscopic patch of space expanded in to a region of $10^{10^{13}}$ and 10^{10^7}(respectively for the quadratic and quartic type of potential inflaton) in a fraction of second. This, in essence, is what basic idea of the theory of inflation.

6.2 Small Field Inflation

The essence of these kind of models is that they don't necessarily require super-Planckian fields. They also predict a different scalar and tensor perturbation (Ref. [7]).

The general type of potential takes the form

$$V(\phi) = V_0 - g\phi^n, n \geq 3. \tag{6.17}$$

We will study the potential

$$V(\phi) = V_0 - \frac{\lambda}{4}\phi^4. \tag{6.18}$$

We note that

$$V' = -\lambda\phi^3, V'' = -3\lambda\phi^2.$$

This gives the slow roll parameters

$$\epsilon = \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{M_{Pl}^2}{16\pi}\left(\frac{-\lambda\phi^3}{V_0 - \frac{\lambda}{4}\phi^4}\right)^2 \approx \frac{M_{Pl}^2}{16\pi}\frac{\lambda^2\phi^6}{V_0^2} \tag{6.19}$$

and similarly

$$\eta = \frac{M_{Pl}^2}{8\pi}\frac{-3\lambda\phi^2}{V_0} \tag{6.20}$$

As we can see, the ϵ and η are less than 1 for small values of ϕ. The inflation can occur if the initial potential value of the inflaton is close to $\phi = 0$.

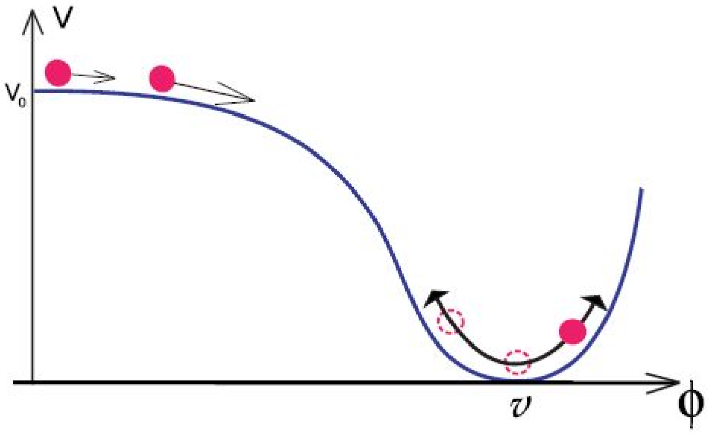


Figure 6.2: The shape of inflaton potential in small field models. The pink circle denotes the evolution of ϕ.

The initial condition required for small field inflation can be explained as follow:

Assuming that prior to inflation the universe was filled with hot material and was uniform temperature throughout. As it is, the effective potential of scalar field is different from its zero-point potential $V(\phi)$. The minimum of the effective potential occurs at $\phi = 0$. Supposing that the inflaton has this property, it's initial value is equals to zero. As the universe cools down due to expansion, th effective potential gradually develops the zero-temperature behavior and the inflation takes place. The point $\phi = 0$is unstable state. Due to quantum effects, the field slightly shifts from zero.

When slow roll ends at the small inflaton field with the condition that

$$\lambda\phi_c^4 << V_0. \tag{6.21}$$

As $\epsilon \sim 1/v_0^2$ and $\eta \sim 1/v_0$, we have $\epsilon << |\eta|$.

The inflation ends when

$$|\eta| = \eta = \frac{M_{Pl}^2}{8\pi}\frac{3\lambda\phi^2}{V_0} \sim 1.$$

This restricts the potential in the following form,

$$\frac{V_0}{M^4{}_{Pl}} << \left(\frac{3}{8\pi}\right)^2 \lambda. \tag{6.22}$$

This is the essence of small field inflaton. Once the inflaton rolls to ϕ_c, it terminates and the reheating occurs.

The relation between the slow roll parameter η and the number of e-fold is given by

$$\eta = \frac{-3}{2N_e}.$$

The general formula for the relation between the slow roll parameter η and the number of e-fold in power-law inflation is given by

$$\eta = -\frac{n-1}{n-2}\frac{1}{N_e}.$$

6.3 Hybrid Inflation

As the name suggests, this kind of models are developed by assuming 2 or more scalar field(one of them is inflaton) drive the inflation. We will study the case of 2 fields, the original field ϕ and one more field χ. The beginning of inflation more or less similar to that of large field inflation. It does not rely on super-Planckian field (Ref. [7]).

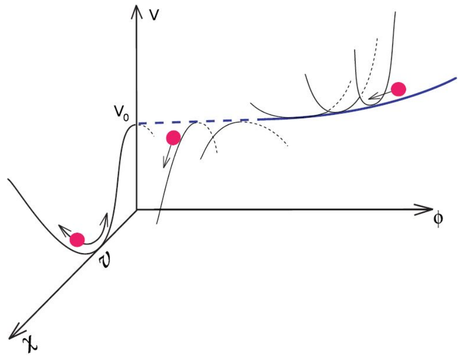


Figure 6.3: The shape of inflaton potential in hybrid model under consideration. The pink circle denotes the evolution of ϕ.

The idea behind the hybrid inflation model is that ϕ is quite large at inflation. It slowly rolls down along $\chi = 0$, inflation ends, the field oscillates near the point $\phi = 0, \chi = v$ and reheating occurs.

For our purpose, we will work with the following potential

$$V(\phi, \chi) = \frac{1}{2}(g^2\phi^2 - \mu^2)\chi^2 + \frac{h}{4}\chi^4 + U(\phi) + V_0. \tag{6.23}$$

Here g and h are taken to be positive dimensionless coupling, μ is a parameter of mass dimension, $U)\phi$ is monotonically increasing inflaton potential. At the point $U(\phi) = 0$, we denote that by V_0. One more assumption is that $U(\phi)$ behaves like power-law potential, such as

$$U(\phi) = \frac{m^2}{2}\phi^2. \tag{6.24}$$

It is noticeable from the shape of the potential form that the potential is minimum at

$$\phi = 0 \ , \chi = v = \frac{\mu}{\sqrt{h}}.$$

The condition that the potential vanishes at the minimum, gives

$$V_0 = \frac{\mu^4}{4h}.$$

Also we see that the point $\chi = 0$is a valley of $V(\phi, \chi)$ for $\phi > \phi_c$. We need the following requirements.

1. The inflation continues till $\phi = \phi_c$.
2. Energy of the inflaton is small in comparison to total energy at the critical point.

Following similar calculation as done previously, we find the slow roll parameter for $\phi > \phi_c$ is

$$\eta(\phi_c) = \frac{M_{Pl}^2}{8\pi}\frac{U''(\phi_c)}{V_0} << 1. \tag{6.25}$$

The other slow roll parameter is

$$\eta(\phi_c) = \frac{M_{Pl}^2}{16\pi}\left(\frac{U'(\phi_c)}{V_0}\right)^2 \sim \eta \cdot \frac{U_c}{V_0} << \eta. \tag{6.26}$$

Now we will find out the time it takes to reach the critical value. In a Hubble time the field shifts by

$$\Delta\phi \sim \dot{\phi}\Delta t \sim \frac{U'(\phi_c)}{3H^2}.$$

The effective potential at this point is

$$V_{eff}(\chi) = V_0 - \frac{\mu_{eff}^2}{2}\chi^2\,, \tag{6.27}$$

with

$$\mu_{eff}^2 = 2g^2\Phi_c\Delta\phi \sim g^2\frac{2U'\phi_c}{3H^2}.$$

Similar to the previous model, the inequality condition for the potential can be put as

$$\frac{(4\pi)^2}{g^2}\frac{V_0}{M_{Pl}^4} \leq \frac{U_c}{V_0} << \frac{\Phi_c^2}{V_0} << 1. \tag{6.28}$$

As it happens, this inequality is satisfied by many parameters, so the hybrid model needs no fine tunning!

Chapter 7

Introduction to Structure Formation

One of the most important contribution of the inflation hypothesis is that it provides a mechanism for the structure formation in the universe. This comes from the inherited quantum fluctuations in ϕ, the inflaton. Inflation tries to make the universe as homogeneous as possible, but the inherited quantum fluctuations does not allows it to do so! After all, our universe is not perfectly homogeneous. The fluctuations in the inflaton(after reheating) leads to density perturbation(also known as scalar perturbation)which eventually leads to structure formation.

As outlined by Liddle in [12], the detailed(and a bit complicated) analysis of the irregularity includes; perturbing the scalar field, expanding the perturbation in comoving wavenumbers, followed by developing the linearized equation for classical evolution, quantization, then finding solution which gives flat space quantum theory($k >> aH$), finding the asymptotic value for($k << aH$) and, finally relating the perturbation in ϕ to the metric/curvature perturbation.

Without going to the detailed mathematical calculation, in the following, we briefly outline the process.

The scalar field $\phi(x)$ can be represented as a Fourier sum as

$$\begin{aligned} \phi(x) &= \sum_{\mathbf{k}=-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t) \\ &= \phi_{k=0}(t) + \sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t) \\ &= \phi_0(t) + \delta\phi(\mathbf{x}, t) \end{aligned} \tag{7.1}$$

In the eq. (2.8), $\phi_0(t)$ is the unperturbed part which depends only on t and $\delta\phi(\mathbf{x}, t)$ is the spatial fluctuation in ϕ, which is a function of both $\mathbf{x}$ and t.

The above expression for ϕ says, due to spatial fluctuations in ϕ, the value of ϕ was not the same everywhere. Hence the energy density associated with ϕ, i.e. $\rho(\phi)$ was not the same everywhere. We can write $\rho_\phi(\mathbf{x})$ as

$$\rho_\phi(\mathbf{x}) = \rho_{\phi_0}(t) + \delta\rho_\phi(\mathbf{x}) \tag{7.2}$$

Now at the end of slow roll, inflation stops and then ϕ oscillates and decays to quarks, leptons, photons, neutrinos, gluons, dark matter, etc. So

$$\rho_{\phi_0} \to \rho_{q,e^-,\gamma,dm}(0) \tag{7.3}$$

$$\delta\rho_{\phi}(\mathbf{x}) \to \delta\rho_{q,e^-,\gamma,dm}(\mathbf{x}) \tag{7.4}$$

Thus the decay of the inhomogeneous inflaton field introduced the inhomogeneity in the matter-energy distribution of the universe. Due to this inhomogeneity, in some region of space there will be an excess of matter compared to its surroundings. Now due to gravitational attraction, this region of space with an excess of matter will attract the matter present in its vicinity, and will ultimately collapse to form structure. Regions that had less matter density initially will develop in to voids.

Thus inflation provides the seed for structure formation.

Chapter 8

Conclusion

We learned that the standard Big-Bang cosmology, i.e the ***Friedmann-Robertson-Walker cosmology*** is quite a successful theory of our universe, in the sense that it explains most of the of the observational facts about our universe, including, but not limited to, the CMB radiation, large scale structure formation, abundance of elements, etc. However, along with the success it enjoys, it raises some profound questions about universe, such as the *horizon* problem, the *flatness* problem, the *monopole* problem, etc. To address these issues, the theory of inflation was brought forward. It postulates a small period of (exponential) metric expansion, which was caused due to a (yet unknown) scalar field ϕ, the inflaton, associated with which was the potential $V(\phi)$, which have the peculiar property that it remains constant, i.e. it does not changes with the metric expansion or time evolution. The inflation hypothesis solves the issues raised by FRW cosmology quite brilliantly.

To make some predictions, which can be verified by experiment/observation the theory requires a particular form of the inflaton. Unfortunately there exist no such standard form of potential. There exist hundreds of models of $V(\phi)$. Each of them makes certain predictions, which can be tested in laboratory or with sky observations. This helps to filter out the appropriate models.

We have studied few such models. We find that the *Higgs* potential (itself) can't drive the inflation. However, it can (non-minimally) couple with gravity to drive the inflation. The concern with this model is that the coupling constant is quite large. We analyzed the *large field* and *small field* models of inflation, calculated few associated relevant parameters. Also, we studied the *hybrid* model, which, unlike the previous models, is a multi-field model. The advantage of this multi-field model is that it does not requires fine-tunning the parameters.

We also briefly outlined how the inherited quantum fluctuations in the inflaton develops irregularity(density perturbation) that provides seed for the struature formation in the universe.

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