INTERVAL FRAME SETS

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AMITA SONI

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Dr. DIVYA SINGH

DEPARTMENT OF MATHEMATICS

NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA

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Certificate

This is to certify that the project report entitled “Interval Frame Sets” submitted by Ms. Amita Soni to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of Master of Science in Mathematics is a record of review work carried out by her under my supervision and guidance. The contents of this project report have not been submitted to any other institute or university for the award of any degree or diploma, to the best of my knowledge.

May 11, 2015

Dr. Divya Singh

Assistant Professor

Department of Mathematics

National Institute of Technology Rourkela

Odisha-769008

Preface

The present thesis consisting of three chapters is devoted to the study of Interval frame sets. After giving the fundamental definitions and results related to basis and frames, we studied frame wavelets. We determined three interval frame sets which are not wavelet sets and their association with frame multiresolution analysis.

Rourkela | Amita Soni

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Amita Soni

M.Sc. Student in Mathematics

Roll No. 413MA2074

National Institute of Technology

Rourkela - 769 008 (India)

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Chapter 1

Preliminaries and Introduction

1.1. Basis in Hilbert Spaces

Bases play an important role in the study of Hilbert spaces. They provide a mean to represent all the elements in a Hilbert space in terms of them. There are several types of basis, like Schauder basis (or, simply basis), orthonormal basis, Riesz basis etc. Orthonormal bases are quite useful because of their efficiency to provide coefficients corresponding to a given vector by simply computing the inner product of the vector with the basis elements.

Definition 1.1.1.[2] A sequence $\{e_k\}_{k=1}^{\infty}$ in Hilbert space H is an *orthonormal system* if $< e_k, e_j >= \delta_{k,j}$. An orthonormal basis is an orthonormal system $\{e_k\}_{k=1}^{\infty}$ which is a basis for H.

From here onwards the symbol H will be used to represent a Hilbert space. The following is a well-known characterization for an orthonormal basis.

Theorem 1.1.2.[2] For an orthonormal system $\{e_k\}_{k=1}^{\infty}$, the following statements are equivalent:

(i) $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis.

(ii) $f = \sum_{k=1}^{\infty} < f, e_k > e_k, \forall f \in H$.

(iii)$< f, g >= \sum_{k=1}^{\infty} < f, e_k >< e_k, g >, \forall f, g \in H.$

(iv) $\sum_{k=1}^{\infty} |< f, e_k >|^2 = \| f \|^2, \forall f \in H.$

(v) $\overline{span}\, \{e_k\}_{k=1}^{\infty} = H.$

(vi) If $< f, e_k >= 0, \forall k \in N$, then $f = 0$.

We know that every separable Hilbert space has an orthonormal basis. The following result given in [2] provides a method to determine orthonormal bases with the help of unitary operators.

Theorem 1.1.3.[2] If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, then the set of all orthonormal bases for H is given by $\{Ue_k\}_{k=1}^{\infty}$, where U is a unitary operator on H.

However, bases have some limitations too. One is due to the lack of flexibility for the choice of coefficients. This limitation can be removed by using frames.

1.2. Frames in Hilbert Spaces

The concept of frame was given by R. J. Duffin and A. C. Schaeffer in their paper, "A class of non-Harmonic Fourier series" in Trans. Amer. Math. Soc. 72 (1952), 341-366.

Definition 1.2.1.[2] A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in H is a *frame* for H if there exist constants $A, B > 0$ such that $A\| f \|^2 \leq \sum_{k=1}^{\infty} |< f, f_k >|^2 \leq B\| f \|^2, \forall f \in H.$

The numbers A and B are known as *frame bounds*. These frame bounds are not unique. When A=B, the frame is said to be *tight frame* and when A=B=1, then it is called a *normalized tight frame*. When an arbitrary element is removed from the sequence and it ceases to be a frame then, the frame is known as an *exact frame*. All these types of frames can be constructed with the help of an orthonormal basis. In fact, if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, then the image of e_k's under any bounded operator onto H determine a frame. Further, as unitary operators on H map an orthonormal basis to another orthonormal basis, similarly they map one frame to another frame, but with the same frame bounds.

In [2] it is proved that for $\{f_k\}_{k=1}^{\infty}$ to be a frame on H, it is sufficient to verify the frame condition on a dense subset of H.

An important example of frames is provided by the Riesz basis. Each Riesz basis is also a frame for H.

Definition 1.2.2. [1] A basis for H is said to a *Riesz basis* if it is equivalent to an orthonormal basis of H. Two bases $\{x_n\}$ and $\{y_n\}$ in H are said to be equivalent if convergence of $\sum a_n x_n$ implies the convergence of $\sum a_n y_n$ and vice-versa, where a_n are scalars.

With each frame an operator can be associated, called the frame operator S, defined by $Sf = \sum_{k=1}^{\infty} < f, f_k > f_k$. A frame can be characterized with the help of frame operator as, $\{f_k\}$ is a frame with frame bounds A and B if and only if the frame operator S is a bounded positive operator with $AI \leq S \leq BI$, where I is the identity map on H. We know that an operator T on H is said to be positive if $< Tx, x > \geq 0$, $\forall x \in H$.

Theorem 1.2.3.[1] Let S be a frame operator of the frame $\{f_k\}$ with frame bounds A and B in H. Then:

(a) S^{-1} exists and positive on H, and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

(b) $\{S^{-1}f_k\}$ is a frame called the dual frame with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$.

(c) Any $f \in H$ can be written in terms of the dual frame.

Proof: (a) From the definition of frame operator it follows that $AI \leq S \leq BI$ and as A, B are non-zero, therefore $I - (1/B)\, S \geq 0$ and $I - (1/B)\, S \leq I - (A/B)\, I$. If T and U are bounded self-adjoint operators on H, then $T \leq U$ implies that $\|T\| \leq \|U\|$, since $\|T\| = sup_{\|x\|=1} |\langle Tx, x\rangle|$. Thus we have, $\| \, I - (1/B)\, S \, \| \leq \| \, I - (A/B)\, I \, \| = 1 - \frac{A}{B} < 1$. Since $(1/B)S$ is close to I, therefore $[(1/B)\, S]^{-1}$ is a bounded operator on all of H and hence so is S^{-1}. Further, for any $x \in H, < S^{-1}x, x > = < S^{-1}x, S\,(S^{-1})\, x > = < S\,(S^{-1})\, x, S^{-1}x > \geq A\| \, S^{-1}x \, \|^2 \geq 0$ shows that S is positive and so self-adjoint (because in Hilbert space all positive operators are self-adjoint). Now S^{-1} is positive and it commutes with S and I. Since, $AI \leq S \Rightarrow S - AI \geq 0 \Rightarrow (S - AI)\, S^{-1} \geq 0 \iff I - AS^{-1} \geq 0 \Rightarrow S^{-1} \leq A^{-1}I$. Also, $S \leq BI \Rightarrow B^{-1}I \leq S^{-1} \leq A^{-1}I$.

(b) As S^{-1} is positive and self-adjoint, so $\sum_{n\in\mathbb{N}} < x, S^{-1}x_n > S^{-1}x_n = \sum_{n\in\mathbb{N}} < S^{-1}x, x_n > S^{-1}x_n = S^{-1}\left(\sum_{n\in\mathbb{N}} < S^{-1}x, x_n > x_n\right) = S^{-1}\left(S\left(S^{-1}x\right)\right) = S^{-1}x$. So, S^{-1} is bounded and linear and $B^{-1}I \leq S^{-1} \leq A^{-1}I$. Therefore, $(S^{-1}x_n)$ is a frame with frame bounds $1/B$ and $1/A$.

(c) $x = SS^{-1}x = S\left(\sum_{n\in\mathbb{N}} < x, S^{-1}x_n > S^{-1}x_n\right) = \sum_{n\in\mathbb{N}} < x, S^{-1}x_n > x_n$ (since S is continuous). Again, take $x = S^{-1}\,(Sx)$. We know that $Sx = \sum_{n\in\mathbb{N}} < x, x_n > x_n$. Substituting this value in x, it becomes $x = S^{-1}\left(\sum_{n\in\mathbb{N}} < x, x_n > x_n\right) \Rightarrow x = \sum_{n\in\mathbb{N}} <$

$x, x_n > S^{-1} x_n$.

Here, the series $f = \sum_{k=1}^{\infty} < f, S^{-1} f_k > f_k, \forall f \in H$ converges unconditionally, i.e. the series converges to the same limit irrespective of the order of appearance of terms in the series.

From the above result it follows that dual frame S^{-1} can be used to determine frame coefficients for a given vector in H. Further, we know that if one vector is removed from an orthonormal basis then it no longer remains a basis, however in case of frame if $< f_j, S^{-1} f_j > \neq 1$ then $\{f_k\}_{k \neq j}$ remains a frame. i.e. the removal of vector f_j from the frame $\{f_k\}_{k=1}^{\infty}$ for H leaves a frame.

Chapter 2

Frame wavelets and Frame sets

2.1. Basics of frame wavelets and frame sets

Definition 2.1.1.[5] A function $\psi \in L^2(\mathbb{R})$ is said to be a *frame wavelet* if $\left\{\psi_{j,k}(s) = 2^{\frac{j}{2}}\psi(2^j s - k); j,k \in \mathbb{Z}\right\}$ forms a frame for $L^2(\mathbb{R})$ i.e there exists two positive constants $A \leq B$ such that for any $f \in L^2(\mathbb{R}), A\| f \|^2 \leq \sum_{j,k\in\mathbb{Z}} |< f, \psi_{j,k} >|^2 \leq B\| f \|^2$. If $A = B$, then ψ is called a normalized tight frame wavelet for $L^2(\mathbb{R})$.

Definition 2.1.2.[5] A measurable set E of $\mathbb{R}$ is called a *frame set* if the Fourier transform of ψ i.e. $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$, is a normalized tight frame wavelet for $L^2(\mathbb{R})$.

Definition 2.1.3. [3] Let E and F be two subsets of $\mathbb{R}$. The set E is said to be translation congruent modulo 2π to F, if there is a measurable partition $\{E_n : n \in \mathbb{Z}\}$ of E such that $\{E_n + 2n\pi : n \in \mathbb{Z}\}$ is a measurable partition of F. Further, E is said to be dilation congruent modulo 2 to F if there is a measurable partition $\{E_n : n \in \mathbb{Z}\}$ of E such that $\{2^n E_n : n \in \mathbb{Z}\}$ is a measurable partition of F.

Theorem 2.1.4.[5] Let E be a measurable subset of $\mathbb{R}$. Then E is a frame set if and only if E is both 2π-translation congruent to a subset F of $[0, 2\pi]$ and 2-dilation congruent to $[-2\pi, -\pi] \cup [\pi, 2\pi]$.

The above result gives a characterization of frame sets. The only difference between the characterization of a frame set and wavelet set is that a wavelet set is 2π-translation congruent to $[0, 2\pi]$, while a frame set is 2π-translation congruent to a subset of $[0, 2\pi]$.

2.2. Two and Three interval frame sets

In [6] all two, three and symmetric four interval wavelet sets were characterized. The following result characterizes all two interval frame sets which are not wavelet sets.

Theorem 2.2.1.[5] Suppose that E is a two-interval set which is not a wavelet set. Then E is a frame set if and only if it has the form $[-2a, -a] \cup [b, 2b]$ with the property that $a, b > 0$ and $a + b \leq \pi$.

Motivated from these results we tried to determine three interval frame sets in this section.

From the second condition that the dilates of a frame set cover the entire real axis it follows that both positive and negative real axis must have at least one of the three intervals. The third interval may be either on the left side, or on the right side of the origin. Further, note that if W is a frame set then so is $-W$ and therefore it is sufficient to construct three interval frame sets with two intervals on the positive real axis and one interval on the negative real axis.

Since a frame set which is not a wavelet set must be 2π translation congruent to a proper subset of $[0, 2\pi]$, therefore in order to construct three interval frame sets we start with three intervals $[a, b)$, $[c, d)$ and $[e, f)$ in $[0, 2\pi]$, where $0 < a < b \leq c < d \leq e < f \leq 2\pi$.

For this choice of intervals first condition of frame sets regarding translation congruence is automatically satisfied. Next these three intervals will be translated on the two sides of the real axis. As per our choice of construction there will be three cases (with two or more subcases) where each of the three intervals will be translated on the negative real axis one by one.

Case I: Suppose that the translate of the interval $[a, b)$ by $2k\pi$, where $k \in \mathbb{N}$, lies on the negative real axis. Then

$$[a, b) - 2k\pi = [2\alpha, \alpha) \cdots\cdots\cdots (1)$$

where α is a negative real number, so that whole negative real axis could be covered by its dilates.

The other two intervals of the required frame set can be taken as $[c, d) + 2l\pi$ and $[e, f) + 2m\pi$, where $l \leq m$ and $l, m \in \mathbb{N} \cup \{0\}$. These two intervals should be related as

$$e + 2m\pi = 2^j (d + 2l\pi) \text{ and } f + 2m\pi = 2^{j+1} (c + 2l\pi) \cdots\cdots (2)$$

with $j \geq 0$.

From (1), we get $b = \frac{a}{2} + k\pi$. As $b < 2\pi$, therefore $k = 1$. Thus

$$b = \frac{a}{2} + \pi. \cdots\cdots\cdots (3)$$

Since $b > \pi$, therefore $\pi < c < d < e < f \leq 2\pi$. From (2), we have $f - e = 2^{j+1}c - 2^j d + 2^{j+1}l\pi = 2^j (2c - d) + 2^{j+1}l\pi$. Since $f - e < \pi$, $2c - d > 0$ and $j \geq 0$, therefore the only possible value of l is zero. By substituting $l=0$ in (2), we get

$$2^j d = e + 2m\pi \text{ and } 2^{j+1} c = f + 2m\pi. \cdots\cdots\cdots (4)$$

Since $\pi < e < 2\pi \Rightarrow \pi + 2m\pi < e + 2m\pi < 2\pi + 2m\pi$, from (4) we obtain

$$\frac{\pi\left(1+2m\right)}{2^j} < d < \frac{2\pi\left(1+m\right)}{2^j}; j \geq 0, m \in \mathbb{N} \cup \{0\}.$$

Again, $\pi < f \leq 2\pi \Rightarrow \pi + 2m\pi < f + 2m\pi < 2\pi + 2m\pi$, and hence

$$\frac{\pi\left(1+2m\right)}{2^{j+1}} < c \leq \frac{2\pi\left(1+m\right)}{2^{j+1}}.$$

Both of the above conditions has to be satisfied simultaneously for $\pi < c < 2\pi$ and $\pi < d < 2\pi$, which is not possible and hence this case does not give any frame set.

Now we will interchange the position of the two intervals on the right side of the origin so that (2) takes the form,

$$2^j\left(f + 2l\pi\right) = c + 2m\pi \text{ and } 2^{j+1}\left(e + 2l\pi\right) \cdots\cdots\cdots (5)$$

where $j \geq 0$, $l, m \in \mathbb{N} \cup \{0\}$ and $l \leq m$. Then, $d - c = 2^j\left(2e - f\right) + 2^{j+1} l\pi$ and again since, $2e - f > 0$, we get $l = 0$.

Therefore (5) becomes, $2^j f = c + 2m\pi$ and $2^{j+1} e = d + 2m\pi$.

Similar to the previous case, this case also does not yield any frame set.

Case II: In this case, instead of $[a, b)$ the middle interval $[c, d)$ is translated to the negative real axis by subtracting $2k\pi$ from it, where $k \in \mathbb{N}$. As in Case I, $[c - 2k\pi, d - 2k\pi) =$

$[2\alpha, \alpha)$, with $\alpha \in \mathbb{R}^-$. Then, $2d = c + 2k\pi$, or

$$d = \frac{c}{2} + \pi \cdots\cdots\cdots (6)$$

since $d < 2\pi$ and hence $k = 1$. This implies that $d > \pi$ and hence, $\pi < e < f \leq 2\pi$.

On the positive real axis we get the relation

$$2^j (b + 2l\pi) = (e + 2m\pi) \text{ and } f + 2m\pi = 2^{j+1} (a + 2l\pi) \cdots\cdots\cdots (7)$$

with $j \geq 0$, $l, m \in \mathbb{N} \cup \{0\}$ and $l \leq m$. Then,

$$f - e = 2^j (2a - b) + 2^{j+1} l\pi < \pi. \cdots\cdots\cdots (8)$$

Now suppose that $c < \pi$, then $0 < a < b < \pi$. Further, assume that $2a - b > 0$. Then in the above relation l should be equal to zero, so that

$$f + 2m\pi = 2^{j+1} a \text{ and } e + 2m\pi = 2^j b. \cdots\cdots\cdots (9)$$

$\pi < e < 2\pi \Rightarrow \pi + 2m\pi < e + 2m\pi < 2\pi + 2m\pi$, or

$$\frac{\pi (1 + 2m)}{2^j} < b < \frac{2\pi (1 + m)}{2^j} \cdots\cdots\cdots (10)$$

Similarly, $\pi < f \leq 2\pi$ gives

$$\frac{\pi (1 + 2m)}{2^{j+1}} < a < \frac{\pi (1 + m)}{2^j} \cdots\cdots\cdots (11)$$

By using the condition $0 < a < b < \pi$, we obtain the relation

$$m < \frac{2^j - 1}{2} \cdots\cdots\cdots (12)$$

Thus, j cannot be zero. For $j = r$, where $r \in \mathbb{N}$, $m \in \{0, 1, 2 \cdots (r-1)\}$. In this case we get frame sets with $0 < a < b < c < \pi < d < e < f \leq 2\pi$, $2a - b > 0$ and conditions given by (6), (9) and (12).

One example of such a frame set is

$$\left[\frac{-11\pi}{6}, \frac{-11\pi}{12}\right) \cup \left[\frac{\pi}{16}, \frac{\pi}{10}\right) \cup \left[\frac{8\pi}{5}, 2\pi\right)$$

where, $a = \frac{\pi}{16}$, $b = \frac{\pi}{10}$, $c = \frac{\pi}{6}$, $d = \frac{13\pi}{12}$, $e = \frac{8\pi}{5}$, $f = 2\pi$, $m = 0$ and $j = 4$.

Next, if $c \geq \pi$ then there will be three cases depending on the values of a and b.

(i) When $0 < a < b < \pi$ and $2a - b > 0$. Here similar to the previous case we will get frame sets. An example of such a frame set is

$$\left[-\pi, -\frac{\pi}{2}\right) \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$$

where, $a = \frac{\pi}{2}$, $b = \frac{3\pi}{4}$, $c = \pi$, $d = \frac{3\pi}{2}$, $e = \frac{3\pi}{2}$, $f = 2\pi$, $m = 0$ and $j = 1$.

(ii) When $0 < a < \pi$, $b \geq \pi$ and $2a - b > 0$. Then from conditions (10) and (11), we get

$$m < \frac{2^{j+1} - 1}{2} \cdots\cdots\cdots (13)$$

In this case we don't get frame sets as $e = 2^j b - 2m\pi \Rightarrow e = 2\left(2^{j-1}b - m\pi\right)$. If we take any value of $b \geq \pi$, this condition is not satisfied. So frame set don't exist in this case.

(iii) When $\pi \leq a < b$. In this case from (10) and (11), we get

$$2^j - 1 < m < 2^j - \frac{1}{2},$$

which is not possible since m is a non-negative integer, therefore in this case we don't get any frame set.

Now interchange the intervals on the positive real axis. So, from the interval on the negative real axis we get the same condition (6) and hence $d > \pi \Rightarrow \pi < e < f < 2\pi$. The relation between the intervals on positive axis becomes

$$2^j(f + 2m\pi) = a + 2l\pi \text{ and } 2^{j+1}(e + 2m\pi) = b + 2l\pi \cdots\cdots\cdots (14)$$

$b - a = 2^{j+1}(e + 2m\pi) - 2^j(f + 2m\pi) = 2^j(2e - f) + 2^{j+1}m\pi.$

Since $2e - f > 0$, therefore $2^j(2e - f) + 2^{j+1}m\pi < \pi \Rightarrow m = 0$.

The relation (14) becomes

$$2^j f = a + 2l\pi \text{ and } 2^{j+1}e = b + 2l\pi \cdots\cdots\cdots (15)$$

Now, $\pi < f < 2\pi \Rightarrow 2^j\pi < 2^j f < 2^{j+1}\pi \Rightarrow 2^j\pi < a + 2l\pi < 2^{j+1}\pi \Rightarrow 2^j\pi - 2l\pi < a < 2^{j+1}\pi - 2l\pi \cdots\cdots\cdots (16)$

Also, $\pi < e < 2\pi \Rightarrow 2^{j+1}\pi < 2^{j+1}e < 2^{j+2}\pi \Rightarrow 2^{j+1}\pi < b+2l\pi < 2^{j+2}\pi \Rightarrow 2^{j+1}\pi - 2l\pi < b < 2^{j+2}\pi - 2l\pi \cdots\cdots\cdots (17)$.

If $c < \pi$, then $0 < a < b < \pi$ so both conditions will not satisfy for any value of l and j simultaneously . So, there exist no frame set in this case.

Next, if $c \geq \pi$, then there will be three cases depending on the values of a and b.

(i) When $0 < a < b < \pi$. Here similar to the previous case we will not get frame sets.

(ii) When $0 < a < \pi$, $b \geq \pi$. Then $\pi < e < f < 2\pi$.

$\pi < f < 2\pi \Rightarrow \pi < \frac{a+2l\pi}{2^j} < 2\pi \Rightarrow 2^j\pi - 2l\pi < a < 2^{j+1}\pi - 2l\pi$ and $\pi < e < 2\pi \Rightarrow \pi < \frac{b+2l\pi}{2^{j+1}} < 2\pi \Rightarrow 2^{j+1}\pi - 2l\pi < b < 2^{j+2}\pi - 2l\pi$.

Since, both the conditions are not satisfied for any value of j and l simultaneously. So, there exist no frame set in this case.

(iii) When $\pi \leq a < b$. In this case from (16) and (17), we get

$$\frac{2^j - 1}{2} < l < \frac{2^{j+1} - 1}{2},$$

which is not possible since l is a non-negative integer, therefore in this case we won't get any frame set.

CASE III: Suppose that the translate of the interval $[e, f)$ by $-2k\pi$, where $k \in \mathbb{N}$ lies on the negative real axis. Then

$[e, f) - 2k\pi = [2\alpha, \alpha) \cdots\cdots\cdots (18)$

where α is a negative real number, so the whole real axis could be covered by the dilates of $[e, f) - 2k\pi$.

The other two intervals of the required frame set can be taken as $[a, b) + 2m\pi$ and $[c, d) + 2l\pi$, where $m < l, m \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$. These two intervals should be related as

$2^j(b + 2m\pi) = c + 2l\pi$ and $2^{j+1}(a + 2m\pi) = d + 2l\pi \cdots\cdots\cdots (19)$ with $j \geq 0$.

From (18), we get $f = \frac{e}{2} + \pi \Rightarrow f > \pi$. From (19) $d - c = 2^j(2a - b) + 2^j m\pi$. Let $(2a - b) > 0$.

If $e < \pi \Rightarrow 0 < a < b < c < d < \pi \Rightarrow d - c < \pi \Rightarrow 2^j (2a - b) + 2^j m\pi < \pi$ and $j \geq 0$, we get $m = 0$.

Therefore (19) becomes,

$2^j b = c + 2l\pi$ and $2^{j+1} a = d + 2l\pi$.

Since, $0 < d < \pi \Rightarrow 2l\pi < d + 2l\pi < \pi + 2l\pi \Rightarrow 2l\pi < 2^{j+1} a < \pi + 2l\pi \Rightarrow \frac{l\pi}{2^j} < a < \frac{\pi(1+2l)}{2^{j+1}}$.

Similarly, $0 < c < \pi \Rightarrow 2l\pi < c + 2l\pi < \pi + 2l\pi \Rightarrow 2l\pi < 2^j b < \pi + 2l\pi \Rightarrow \frac{2l\pi}{2^j} < b < \frac{\pi(1+2l)}{2^j}$

For l=r and j$\geq r + 1$, these two conditions satisfy simultaneously. Hence, we get frame sets in this case.

One example of such a frame set is

$$\left[\frac{-19\pi}{18}, \frac{-19\pi}{36}\right) \cup \left[\frac{17\pi}{96}, \frac{\pi}{3}\right) \cup \left[\frac{8\pi}{3}, \frac{17\pi}{6}\right),$$

where $a = \frac{17\pi}{96}, b = \frac{\pi}{3}, c = \frac{2\pi}{3}, d = \frac{5\pi}{6}, e = \frac{17\pi}{18}, f = \frac{53\pi}{36}, l = 1, j = 3$.

If $e \geq \pi$,

$c > \pi \Rightarrow \pi < c < d < e < f < 2\pi$.

(i) When $0 < a < b < \pi$,

$2^j (b + 2m\pi) = c + 2l\pi$ and $2^{j+1} (a + 2m\pi) = d + 2l\pi \cdots\cdots\cdots (20)$.

From (21), we get m=0 as $d - c < \pi$. Also, (20) becomes

$2^j b = c + 2l\pi$ and $2^{j+1} a = d + 2l\pi$.

Now, $\pi < c < 2\pi \Rightarrow \pi + 2l\pi < c + 2l\pi < 2\pi + 2l\pi \Rightarrow \frac{\pi(1+2l)}{2^j} < b < \frac{2\pi(1+l)}{2^j} \cdots\cdots\cdots (21)$.

Similarly, $\pi < d < 2\pi \Rightarrow \pi+2l\pi < d+2l\pi < 2\pi+2l\pi \Rightarrow \frac{\pi+2l\pi}{2^{j+1}} < a < \frac{2\pi+2l\pi}{2^{j+1}} \cdots\cdots\cdots (22)$.

From here we get, for $l = r, j \geq r + 1$, frame set exist. An example can be given by

$$\left[\frac{-7\pi}{9}, \frac{-7\pi}{18}\right) \cup \left[\frac{19\pi}{96}, \frac{7\pi}{18}\right) \cup \left[\frac{28\pi}{9}, \frac{19\pi}{6}\right),$$

where $a = \frac{19\pi}{96}, b = \frac{7\pi}{18}, c = \frac{10\pi}{9}, d = \frac{7\pi}{6}, e = \frac{11\pi}{9}, f = \frac{29\pi}{18}, l = 1, j = 3$.

(ii) When $a < \pi$ and $b \geq \pi$. Then (21)and (22) are satisfied only for j=r and l=r, for r$\in \mathbb{N} \cup \{0\}$.

One example of such a frame set is

$$\left[\frac{-\pi}{3}, \frac{-\pi}{6}\right) \cup \left[\frac{49\pi}{72}, \frac{4\pi}{3}\right) \cup \left[\frac{16\pi}{3}, \frac{49\pi}{9}\right),$$

where $a = \frac{49\pi}{72}, b = \frac{4\pi}{3}, c = \frac{4\pi}{3}, d = \frac{13\pi}{9}, e = \frac{5\pi}{3}, f = \frac{33\pi}{18}, l = 2, j = 2$.

(iii) When $\pi < a < b < 2\pi$, then condition (21) and (22) not satisfied simultaneously so there exists no frame set in this case.

When $d < \pi \Rightarrow 0 < a < b < c < d < \pi$ and $\pi \geq e < f < 2\pi$.

From (20), we get m=0 as $d - c < \pi$. Also, (20) becomes $2^j b = c + 2l\pi$ and $2^{j+1}a = d + 2l\pi \cdots\cdots\cdots (23)$.

Since, $0 < d < \pi \Rightarrow 2l\pi < 2^{j+1}a < \pi + 2l\pi \Rightarrow \frac{l\pi}{2^j} < a < \frac{\pi+2l\pi}{2^{j+1}} \cdots\cdots\cdots (24)$.

Similarly, $0 < c < \pi \Rightarrow 2l\pi < 2^j b < \pi + 2l\pi \Rightarrow \frac{2l\pi}{2^j} < b < \frac{\pi(1+2l)}{2^j} \cdots\cdots\cdots (25)$.

For l=r,$j \geq r + 1$, (24) and (25) satisfied simultaneously. So, frame set exist in this case.

One example of such a frame set is

$$\left[\frac{-\pi}{3}, \frac{-\pi}{6}\right) \cup \left[\frac{5\pi}{24}, \frac{59\pi}{144}\right) \cup \left[\frac{59\pi}{9}, \frac{20\pi}{3}\right),$$

where $a = \frac{5\pi}{24}, b = \frac{59\pi}{144}, c = \frac{5\pi}{9}, d = \frac{2\pi}{3}, e = \frac{5\pi}{3}, f = \frac{33\pi}{18}, l = 3, j = 4.$

Chapter 3

Frame multiresolution analysis and Superwavelets

3.1. Frame Multiresolution Analysis

Similar to the concept of Multiresolution Analysis (MRA) for wavelets, Frame Multiresolution Analysis (FMRA) is defined for frame wavelets. Among the five conditions, first four conditions for MRA and FMRA are same. Only difference is in the last condition corresponding to the scaling function. In MRA the set of integer translates of ϕ forms an orthonormal basis for V_0, while in FMRA the same set forms a frame for V_0.

Definition 3.1.1.[2] Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces in $L^2(\mathbb{R})$, then it forms a *FMRA* if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$;

(ii) $f(.) \in V_j$ if and only if $f(2.) \in V_{j+1}, \forall j \in \mathbb{Z}$;

(iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iv) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;

(v) There exist a function $\phi \in V_0$ such that $\{T_k\phi\}_{k\in Z}$ is a frame for V_0.

Theorem 3.1.2.[4] Let E be a bounded frame set. Then ψ_E is an FMRA wavelet if and only if E^s is 2π-translation congruent to a subset of $[-\pi, \pi)$, where $E^s = \bigcup_{k\geq 1} 2^{-k}E$ for a given set $E \subseteq \mathbb{R}$.

By using the above result we can easily determine that whether frame wavelets determined by the three interval frame sets given above are associated with an FMRA or, not. For example consider the frame set

$$E = \left[-\frac{11\pi}{6}, -\frac{11\pi}{12} \right) \cup \left[\frac{\pi}{8}, \frac{\pi}{6} \right) \cup \left[\frac{4\pi}{3}, 2\pi \right)$$

Then

$$E^S = [-\frac{11\pi}{12}, 0) \cup [0, \frac{\pi}{8}) \cup [\frac{\pi}{6}, \frac{\pi}{4}) \cup [\frac{\pi}{3}, \frac{\pi}{2}) \cup [\frac{2\pi}{3}, \pi),$$

which is clearly 2π-translation congruent to a subset of $[-\pi, \pi)$ and hence the normalized tight frame wavelet determined by this frame set is associated with an FMRA.

Similarly, for

$$E = \left[-\frac{\pi}{3}, -\frac{\pi}{6} \right) \cup \left[\frac{49\pi}{72}, \frac{4\pi}{3} \right) \cup \left[\frac{16\pi}{3}, \frac{49\pi}{9} \right)$$

$$E^s = \left[\frac{-\pi}{6}, \frac{2\pi}{3} \right) \cup \left[\frac{8\pi}{3}, \frac{49\pi}{18} \right),$$

is again 2π-translation congruent to a subset of $[-\pi, \pi)$.

In the same way for all other examples which we have determined above, it can be shown that they are associated with an FMRA.

3.2. Superwavelets

One of the most important factor behind the popularity of wavelets in signal processing is MRA. However, if we want to transmit more than one signals at a time then because of the reason given below it is preferable to use superwavelets instead of wavelets.

Let $\{x_n\}_{n\in J}$ and $\{y_n\}_{n\in J}$ be frames in Hilbert spaces H and K, respectively. These two frames are said to be strongly disjoint if $\{x_n \oplus y_n \mid n \in J\}$ is a frame for the Hilbert space $H \oplus K$.

The operations vector addition, scalar multiplication and inner product in direct sum of Hilbert spaces $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus ... \oplus L^2(\mathbb{R})$ (n times) is defined by

(i) $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$.

(ii) $\alpha(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$.

(iii) $<(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)>= (< x_1, y_1 >, < x_2, y_2 >, ..., < x_n, y_n >)$,

where $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus ... \oplus L^2(\mathbb{R})$.

Strong disjointness of two frames $\{x_n\}$ and $\{y_n\}$ can be characterized by the following two equivalent conditions:

$\sum_{n\in J} < x, x_n > y_n = 0$, $\forall x \in H$; or $\sum_{n\in J} < y, y_n > x_n = 0$, $\forall y \in K$.

For a strongly disjoint pair of frames one set of coefficients is sufficient to represent two vectors x and y in H and K, respectively. This idea is helpful for data compression and can be easily extended to the m-tuple case.

Definition 3.2.1.[4] Let $\psi_1, \psi_2, ..., \psi_m$ be normalized tight frame wavelets in $L^2(\mathbb{R})$. The m-tuple $(\psi_1, \psi_2, ..., \psi_m)$ is called a *super-wavelet* in the super space $L^2(\mathbb{R}) \oplus ... \oplus L^2(\mathbb{R})$ (added m-times) if the set $\left\{\left(D^n T^l \psi_1, D^n T^l \psi_2, ..., D^n T^l \psi_m\right) : n, l \in \mathbb{Z}\right\}$ is an orthonormal basis for the super-space.

The following result gives a method of determining superwavelets with only two components.

Theorem 3.2.2:[4] Let E and F be two frame sets then (ψ_E, ψ_F) is a super-wavelet if and only if for any $k \in \mathbb{Z}$, we have $(E + 2k\pi) \cap F = \phi$ and $E \cup F$ is $2\pi-$translation congruent to $[0, 2\pi)$.

In the procedure given above to determine three interval frame sets we started with three-intervals lying inside $[0, 2\pi)$ and these three intervals were then translated on the positive and negative real line by integer multiples of 2π. Therefore by simply looking at these three-intervals inside $[0, 2\pi)$ we can say whether a given pair of three-interval frame sets will generate a superwavelet, or not.

For example the following two frame sets cannot generate a superwavelet.

$$E = \left[\frac{-19\pi}{18}, \frac{-19\pi}{36}\right) \cup \left[\frac{17\pi}{96}, \frac{\pi}{3}\right) \cup \left[\frac{8\pi}{3}, \frac{17\pi}{6}\right)$$

$$F = \left[\frac{-\pi}{3}, \frac{-\pi}{6}\right) \cup \left[\frac{49\pi}{72}, \frac{4\pi}{3}\right) \cup \left[\frac{16\pi}{3}, \frac{49\pi}{9}\right)$$

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