Some studies on infinite-dimensional Lie(super) algebras

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This is to certify that the work presented in this dissertation entitled *"Some studies on infinite-dimensional Lie(super) algebras"* by *Saudamini Nayak*, Roll Number 511MA101, is a record of original research carried out by her under my supervision and guidance in partial fulfillment of the requirements of the Doctor of Philosophy in Mathematics. Neither this dissertation nor any part of it has been submitted for any degree or diploma to any institute or university in India or abroad.

Kishor Chandra Pati

Dedicated

to my

Loving Parents Mr. Soubhagya Ch. Nayak

and Mrs. Shakuntala Nayak

Declaration of Originality

I, *Saudamini Nayak*, Roll Number 511MA101 hereby declare that this dissertation entitled *"Some studies on infinite-dimensional Lie(super) algebras*" represents my original work carried out as a doctoral student of NIT Rourkela and, to the best of my knowledge, it contains no material previously published or written by another person, nor any material presented for award of any other degree or diploma of NIT Rourkela or any other institution. Any contribution made to this research by others, with whom I have worked at NIT Rourkela or elsewhere, is explicitly acknowledged in the dissertation. Works of other authors cited in this dissertation have been duly acknowledged under the section Bibliography. I have also submitted my original research records to the scrutiny committee for the evaluation of my dissertation.

I am fully aware that in case of any non-compliance detected in future, the Senate of NIT Rourkela may withdraw the degree awarded to me on the basis of present dissertation.

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Abstract

In this thesis, we study some results on infinite dimensional Lie algebras. Total thesis is divided into three parts, i.e., on first part we have determined untwisted affine Kac-Moody symmetric spaces, second part is devoted towards embedding of hypebolic Kac-Moody superalgberas and in the final part we study some branching laws for certain infinite dimensional reductive pair of Lie algebras.

Symmetric spaces associated with Lie algebras and Lie groups which are Reimannian manifolds have recently got a lot of attention in various branches of physics and mathematics. Their infinite dimensional counterpart have recently been discovered which are affine Kac-Moody symmetric spaces. We have (algebraically) explicitly computed the affine Kac-Moody symmetric spaces associated with affine Kac-Moody algebras $A_1^{(1)}, A_2^{(1)}$ and $A_2^{(2)}$. We have also computed all the affine untwisted Kac-Moody symmetric spaces starting from the Vogan diagrams of the affine untwisted classical Kac-Moody Lie algebras.

Root systems and Dynkin diagrams play a vital role in understanding and explaining the structure of corresponding algebras and superalgebras. Here through the help of the Dynkin diagrams and root systems we have given a super symmetric version of a theorem by S. Viswanath for hyperbolic Kac-Moody superalgebras. We have shown that $HD(4,1)$ hyperbolic Kac-Moody superalgbera of rank 6 contains every simplylaced Kac-Moody subalgebra with degenerate odd root as a Lie subalgebra.

Branching law is a classical problem in the representation theory of finite dimensional Lie algebras. Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{g}'$ be the Lie subalgebra of $\mathfrak{g}$ and V be irreducible $\mathfrak{g}$-module then, V is no longer an irreducible $\mathfrak{g}'$-module. A branching law amounts to a decomposition of V into irreducible $\mathfrak{g}'$-module. However such a decomposition does not exist necessarily. The branching laws are understandable to some extent, in some nice setting (when $\mathfrak{g}$ and $\mathfrak{g}^{'}$ are semisimple and V is finite dimensional). But for classical pairs $(\mathfrak{g}, \mathfrak{g}^{'})$ such as $(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$, $(\mathfrak{so}_n, \mathfrak{so}_{n-1})$ etc. branching laws are explicitly known. Since each classical Lie algebra $\mathfrak{g}$ fits into a descending family of classical algebras, the irreducible representations of $\mathfrak{g}$ can be studied inductively. Here we have studied some branching laws for certain pairs $(\mathfrak{g}, \mathfrak{g}')$ of infinite dimensional Lie algebras which are inductive limit of finite dimensional reductive Lie algebras.

Keywords: Kac-Moody group; Kac-Moody algebra; Tame Fréchet manifold; Affine Kac-Moody symmetric space; Hyperbolic Kac-Moody superalgebra; Embedding; Direct limit; Branching law.

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Chapter 1

Introduction

Root systems and Dynkin diagrams play an important role in understanding structure theory (real forms and symmetric spaces, embedding) and representation theory (highest weight representations, branching law) of finite dimensional complex simple Lie algebras as well as for infinite dimensional Lie algebras and Lie superalgebras. This has become possible due to the fact that Vogan diagrams/ Satake diagrams of these algebras have already been classified. These diagrams are nothing but modified Dynkin diagrams with some extra piece of information and the weights in adjoint representations are roots. In our thesis we have used these mathematical tools to study some aspects of structure theory and representation theory involving infinite dimensional Lie algebras and superalgebras.

Building on the late nineteenth century researches of Sophus Lie and Wilhelm Killing, Eli Cartan completed the classification of the finite dimensional simple Lie algebras (FSLA) over the complex numbers $\mathbb{C}$ [Car94]. Killing and Cartan [Hum72] derived the fundamental system of simple roots associated with Lie algebras by using generalized eigenspace decomposition and the classifications of finite dimensional irreducible representation of FSLA over $\mathbb{C}$ [Car13] was carried out by Cartan involving the weight space decomposition. Weyl provided the additional level of familiarity to this; leading to his so called character formula [Wey26]. This had further natural extension over other fields such as the real field $\mathbb{R}$, the number fields as pointed out by different researchers like A. Albert, H. Freudenthal, N. Jacobson, G. Seligman, J. Tits and Cartan [Car14]. But the most successful and penetrating theory was issued from the work of C. Chevalley [Che48] and Harish-Chandra [HC51] who indicated a way to construct simultaneously the FSLA's and all of the their finite dimensional irreducible representations. Also Cartan had established that there was one and only one simple Lie algebra corresponding to each of nine types of finite Cartan matrices. This made Chevalley to single out the elegant construction Ernest Witt [Wit41] had given, showing the existence of the five exceptional types of algebras.

Like Chevalley [Che48] and Harish Chandra [HC51] using the ideas of Cartan, Weyl gave the representation theory of semisimple Lie algebras over the field of complex numbers, what is now called as Harish-Chandra homomorphisms. This paper became one of the foundational pillars of analysis on semisimple Lie groups [Her91].

This theory was further simplified and extended to certain infinite dimensional Lie algebras by N. Jacobson in his text book Lie algebras [Jac79]. The line of research of Chevalley and Harish-Chandra came to a natural conclusion when Jean-Pierre Serre gave a presentation [Ser66] for all FSLA's over $\mathbb{C}$, a result known as Serre's theorem. This marked a natural beginning for a very prominent theme in both mathematics and physics, namely the theory of Kac-Moody algebras.

Beginning in the mid 1960, Robert Moody in Canada and Victor G. Kac in Russia worked simultaneously and independently to extend the construction of Jacobson [Jac79] to the infinite dimensional setting to classify and represent what is known today as the Kac-Moody Lie algebras. This construction represented by no means the main thrust of their earlier work, both singled out the particular subclass of algebras now termed as the affine Kac-Moody Lie algebras giving realizations and obtaining deep structural information about them.

Both Kac [Kac67, Kac68a, Kac68b] and Moody [Moo67, Moo69] followed their initial series of papers with some additional research on their new class of algebras. While Kac published papers like *Some properties of contragradient Lie algebras* [Kac69b] and *Automorphisms of finite order semisimple Lie algebras* [Kac69a], Moody [Moo70] on the other hand analyzed *Simple quotients of Euclidean Lie algebras*. Both also worked on other Lie theoretic topics.

The immediate natural context of this new class of algebra came when Ian Macdonald published a paper *Affine root systems and Dedkind's η-function* [Mac72], where he noted that the classification of affine root system that he presented in his paper was similar to that given by Moody [Moo68, Moo69] in the case of Euclidean Lie algebras (infinite dimensional Lie algebras).

Recognizing the importance of their new class of algebras, both Kac and Moody made further discoveries independently. While Kac submitted a paper *Infinite dimensional Lie algebras and Dedkind's η-function* [Kac74], Moody on the other hand worked on *Macdonald's identities and Euclidean Lie algebras* [Moo75].

So, the algebra that Kac and Moody had discovered, attracted increasing attention following their linkage to MacDonald's results. Beyond this, there are other profound connections with other topics in mathematics such as finite simple groups, areas of topology and cohomology [GL76]. So the algebra not only defines a vibrant and burgeoning sub field of mathematics with surprising applications but also defines an area of spectacular growth and influence in physics [Lou95].

Over the past years it has become clear that infinite dimensional Lie algebras play an increasingly important role in modern theoretical physics, string theory in particular. The close links between string operators and Kac-Moody algebras are well known. Also, there has been mounting evidence that Kac-Moody algebras of indefinite type and generalized Kac-Moody algebras might appear in the guise of duality symmetries in string and M-theory. Significant application of affine Kac-Moody algebra has also been appeared in the theory of heterotic strings. As the subject of study is very vast, in this thesis we confine ourselves to a tiny fraction of it.

The connection between random matrix theory and symmetric spaces is obtained simply through the coset spaces defining the symmetry classes of random matrix ensembles. Although Dyson was the first to recognize that the coset spaces are symmetric spaces, the subsequent emergence of new random matrix symmetry classes and their classification in terms of Cartan's symmetric spaces is relatively recent. According to Cartan, there exist 11 large families of symmetric spaces. Those of type II are compact, unitary, orthogonal and sympletic Lie groups (A,B,C,D). The large families of type I symmetric spaces are denoted by $AI, AII, AIII, BDI, CI, CII$ and $DIII$. The standard Wigner-Dyson classes of a mass less Dirac particle derive from $AIII$(Chiral GUE), BDI(Ch GOE) and AII(Ch GSE).

Now it is beyond doubt that Kac-Moody algebras [Kac68b, Moo67] more particularly, the affine version have wide physical applications in the context of integrable systems, two dimensional field theories and string theories etc., whose importance lies on the same footing as that of ordinary Lie algebras. Thus the potential physical applications of affine Kac-Moody algebras/ affine Kac-Moody symmetric spaces in the field of quantum electronic transport and new type of random matrix ensembles cannot be denied.

Kac-Moody algebras are of three types finite, affine and indefinite. Further, affine type is divided into two subclasses namely affine untwisted and affine twisted Kac-Moody algebras. A subclass of indefinite type algebra is hyperbolic Kac-Moody algebra. The hyperbolic Kac-Moody algebras exist only in ranks $2-10$ and have already been classified.

In this thesis we have determined the real forms associated with affine untwisted Kac-Moody algebras (constructed from the so called classical Lie algebras) by Vogan diagrams and algebraic methods and finally we have constructed the affine Kac-Moody symmetric spaces associated with the algebras.

After these types studies related to affine Kac-Moody algebras, we then switch over to the case of supersymmetric version of Kac-Moody algebra, i.e., Kac-Moody superalgebra. Similar to Kac-Moody algebras, Kac-Moody superalgebras are also divided into three categories namely finite, affine and indefinite type. A subclass of indefinite type is hyperbolic Kac-Moody superalgebra. Here we want to point out that the hyperbolic Kac-Moody superalgebras exist only in rank 2-6. The hyperbolic Kac-Moody superalgebras are finite in number, with rank > 2 with maximum rank being 6.

In a paper [Vis08] Viswanath has shown that hyperbolic Kac-Moody algebra E_{10} contains every simplylaced hyperbolic Kac-Moody algebra as a Lie subalgebra. In super case there are exactly 17 simplylaced Kac-Moody superalgebra in rank 3-6 [CCL10, FS05, TDP03]. Here we give the supersymmetric version of the result by Viswanath, showing that the hyperbolic Kac-Moody superalgebra $HD(4,1)$ of rank 6 contains every simplylaced Kac-Moody subalgebra with degenerate odd root, as a Lie subalgebra.

After some studies on infinite dimensional Lie algebras of affine untwisted Kac-Moody algebras and hyperbolic Kac-Moody superalgebras types where, primarily the root systems of these algebras are extensively exploited for such studies, now we turn to altogether a different type of study, i.e., branching laws for infinite dimensional reductive Lie algebras with some conditions on weight.

A classical problem in the representation theory of a compact Lie group G is to calculate the restriction of an irreducible representation of G to a closed subgroup K, is called branching problem. Similarly, for the corresponding Lie algebra $\mathfrak{g}$, of the Lie group G one can think of the same problem. Precisely, for a pair $(\mathfrak{g}, \mathfrak{g}')$ where $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ and an irreducible $\mathfrak{g}$-module V, the branching problem is to determine the structure of V as $\mathfrak{g}'$-module. When $\mathfrak{g}$ and $\mathfrak{g}'$ are semisimple and V is finite dimensional, the branching problem reduces to finding the multiplicity of any simple $\mathfrak{g}'$-module V' as direct summand of V. Even in this nice setting (in particular for classical series of Lie algebras) also, an explicit solution for branching problem is known only for specific cases.

Since each classical Lie algebra $\mathfrak{g}$ fits into a descending family of classical algebras, the irreducible representations of $\mathfrak{g}$ can be studied inductively which gives rise to the branching problem. Here we have studied branching law for a pair $(\mathfrak{g}, \mathfrak{g}')$ of infinite dimensional Lie algebras which are direct limit of finite dimensional reductive Lie algebras.

The mode of representation of the thesis is as follows. In Chapter 2, in order to make the reader familiar, we present the notations, preliminaries associated with Kac-Moody algebras. Then we emphasize on root systems, real forms, Vogan diagrams of FSLA's along with symmetric spaces associated with it. We also explicitly shown the root system associated with affine (untwisted and twisted) Kac-Moody algebras. At the end of the Chapter we show explicitly the way of algebraic calculation of symmetric spaces related to A_1 and A_2.

In Chapter 3, we give a realization of affine untwisted Kac-Moody algebra. We have also given a brief introduction to the involution, real forms, Vogan diagrams, affine Kac-Moody symmetric spaces associated with affine untwisted Kac-Moody algebras. We have exclusively determined the real forms associated with some low rank affine Kac-Moody algebras like $A_1^{(1)}, A_2^{(1)}$ and $A_2^{(2)}$ and also determined the affine Kac-Moody symmetric spaces associated with these algebras and relating them with their Vogan diagrams to corroborate our technique to construct affine Kac-Moody symmetric spaces.

In Chapter 4, we give an introduction to the definition and structure of hyperbolic Kac-Moody superalgebras and their Dynkin diagrams. Root systems and Dynkin diagrams play a vital role in understanding and explaining the structure of corresponding algebras and superalgebras. Here through the help of the Dynkin diagrams and root systems we give a supersymmetric version of a theorem by S. Viswanath for hyperbolic Kac-Moody superalgebras. Particularly, we have shown that the hyperbolic Kac-Moody superalgebra $HD(4,1)$ of rank 6 contains every simplylaced Kac-Moody subalgebra with degenerate odd root, as a Lie subalgebra.

In chapter 5, starting from representation of complex semisimple Lie algebra $\mathfrak{g}$ on a vector space V we have defined highest weight, highest weight vector and highest weight module of $\mathfrak{g}$ and stated the highest weight theorem. Also we give the statement of the Weyl branching law for the Lie algebra $\mathfrak{gl}(n)$. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be reductive where $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ generated by $\Pi' \subset \Pi$ where Π is the simple root system of $\mathfrak{g}$. From highest weight theorem we know that to each dominant integral weight λ there is an unique irreducible highest weight representation say V_λ. Consider $\lambda_\infty = (\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_m, \ldots)$ (i.e. λ_∞ is bounded) such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m = \lambda_m, \cdots$ is an arbitrary highest weight with highest weight module V_{λ_∞}. For $\mathfrak{g} = \mathfrak{g}(\infty)$(direct limit of $\mathfrak{g}(n)$ with natural embedding maps)-module V_{λ_∞}, branching law is given. Before that we have given some concrete examples for different $\mathfrak{g} = \mathfrak{gl}(\infty)$-module V.

Chapter 2

Notations and Preliminaries

2.1 Kac-Moody algebra

Let $I = [1,n], n \in \mathbb{N}$, be an interval in $\mathbb{N}$. We start with a complex $n \times n$ matrix $A = (a_{ij})_{i,j \in I}$ of rank l and we will associate with it a complex Lie algebra $\mathfrak{g}(A)$.

Definition 2.1.1. *We call A, a generalized Cartan matrix (GCM) if it satisfies the following conditions:*

1. $a_{ii} = 2$ *for* $i = 1, \ldots, n$*;*
2. $a_{ij} \leq 0$ *for* $i \neq j$*;*
3. $a_{ij} = 0$ *iff* $a_{ji} = 0$*.*

2.1.1 Realization of a matrix

A realization of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, satisfying the conditions:

1. both sets Π and $\Pi^\vee$ are linearly independent;
2. $< \alpha_i^\vee, \alpha_j >= a_{ij}$ for $i, j = 1, 2, \ldots, n$;
3. $n - l = \dim \mathfrak{h} - n$.

Two realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ are said to be isomorphic if there exists vector space isomorphism $\phi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that$\phi(\Pi_1^\vee) = \Pi_2^\vee$ and $\phi^*(\Pi_2) = \Pi_1$. It is well known that realization exists and is unique upto isomorphism [Kac90].

2.1.2 Construction of the auxiliary Lie algebra

Let A be a complex $n \times n$ matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ be its realization. Then the auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ is generated by a basis of the standard Cartan subalgebra $\mathfrak{h}$ and the elements e_i, f_i for $i \in I$ with the following defining relations [Kac90]:

$$[e_i, f_j] = \delta_{ij} h_i \tag{2.1.1}$$

$$[h, h'] = 0 \tag{2.1.2}$$

$$[h, e_i] = a_{ij} e_i \tag{2.1.3}$$

$$[h, f_i] = -a_{ij} f_i \tag{2.1.4}$$

where $h, h' \in \mathfrak{h}$ and the Serre relations

$$(ad\ e_i)^{1-a_{ij}} e_j = 0, \quad (ad\ f_i)^{1-a_{ij}} f_j = 0; \ \forall i \neq j. \tag{2.1.5}$$

The uniqueness of the realization guarantees that $\tilde{\mathfrak{g}}(A)$ depends only on A. Denote the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, e_2, \ldots, e_n$ (resp. $f_1, f_2, \ldots, f_n$) by $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$). We now have the following properties of the auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$.

Theorem 2.1.2. *[Kac90]*

1. $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$.
2. $\tilde{\mathfrak{n}}_+$ *(resp.* $\tilde{\mathfrak{n}}_-$*) is freely generated by* $e_1, e_2, \ldots, e_n$ *(resp.* $f_1, f_2, \ldots, f_n$*).*
3. *The map* $e_i \mapsto -f_i, f_i \mapsto -e_i (i \in I), h \mapsto h (h \in \mathfrak{h})$ *can be uniquely extended to an involution of the Lie algebra* $\tilde{\mathfrak{g}}(A)$.
4. *With respect to* $\mathfrak{h}$ *one has the root space decomposition*

$$\tilde{\mathfrak{g}}(A) = (\bigoplus_{\alpha \neq 0} \tilde{\mathfrak{g}}_{-\alpha}) \oplus \mathfrak{h} \oplus (\bigoplus_{\alpha \neq 0} \tilde{\mathfrak{g}}_{\alpha})$$

   where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) : [h, x] = \alpha(h) x\}$.
5. *Among the ideals of* $\tilde{\mathfrak{g}}(A)$ *intersecting* $\mathfrak{h}$ *trivially, there exists a unique maximal ideal* τ. *Furthermore*

$$\tau = (\tau \cap \tilde{\mathfrak{n}}_-) \oplus (\tau \cap \tilde{\mathfrak{n}}_+).$$

2.1.3 Construction of the Kac-Moody algebra

Definition 2.1.3. *Let A be a generalized Cartan matrix and* $\tilde{\mathfrak{g}}(A)$ *be its auxiliary Lie algebra. By the previous theorem, there exist an embedding* $\mathfrak{h} \to \tilde{\mathfrak{g}}(A)$. *Let* τ *be the maximal ideal intersecting* $\mathfrak{h}$ *trivially i.e.,* $\tau \cap \mathfrak{h} = 0$. *Define,*

$$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\tau.$$

The Lie algebra $\mathfrak{g}(A)$ *is called the Kac-Moody algebra associated with GCM A, and n is called the rank of* $\mathfrak{g}(A)$.

Definition 2.1.4. *The center of the Lie algebra* $\mathfrak{g}(A)$ *is*

$$\mathfrak{c} := \{h \in \mathfrak{h} :< \alpha_i, h >= 0 \ \forall \ i = 1, 2 \ldots, n\}.$$

Further, $\dim \mathfrak{c} = n - l$ *where* l *is rank of* A.

2.1.4 Root space of the Kac-Moody algebra

Let $\mathfrak{g}(A)$ be a Kac–Moody algebra with Cartan subalgebra $\mathfrak{h}$ and root data Δ. The root space $\mathfrak{g}_\alpha$ is defined by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) : [h, x] = \alpha(h)x \ \ \forall h \in \mathfrak{h}\}.$$

Then we have the root space decomposition

$$\mathfrak{g}(A) = (\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha) \oplus \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha),$$

which is a decomposition of $\mathfrak{g}(A)$ into finite dimensional subspaces, where Δ_+ (resp. Δ_-) is the set of positive (resp. negative) roots. The dimension of the root space $\mathfrak{g}_\alpha$ is called the multiplicity of α. Root multiplicities are fundamental data to understand the structure of a Kac– Moody algebra $\mathfrak{g}(A)$.

Definition 2.1.5. *For each* $i = 1, 2, \ldots, n$*, we define the fundamental reflection* r_i *of the space* $\mathfrak{h}^*$ *by*

$$r_i(\lambda) = \lambda - < \lambda, \alpha_i^\vee > \alpha_i, \ \ \alpha_i \in \mathfrak{h}^*. \tag{2.1.6}$$

Here r_i is indeed a reflection, as its fixed point set is $T_i = \{\lambda \in \mathfrak{h}^* :< \lambda, \alpha_i^\vee >= 0\}$, and $r_i(\alpha_i) = -\alpha_i$.

Definition 2.1.6. *The subgroup* $W \subset GL(\mathfrak{h}^*)$ *generated by all fundamental reflections is called the Weyl group* W *of* $\mathfrak{g}(A)$.

From this definition we know that the root system Δ of $\mathfrak{g}(A)$ is W-invariant, and $\text{mult}\ \alpha = \text{mult}\ w(\alpha), \forall \alpha \in \Delta, w \in W$.

Definition 2.1.7. *A root* $\alpha \in \Delta$ *is called real if there exists a* $w \in W$ *such that* $w(\alpha)$ *is a simple root.*

We denote the set of all real roots by Δ^{re} and the set of all positive real roots by Δ_+^{re}. Given any $\alpha \in \Delta^{\text{re}}$, then $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Pi, w \in W$.

Definition 2.1.8. *A root* $\alpha \in \Delta$ *is called imaginary if it is not real.*

We denote the set of all imaginary roots by Δ^{im} and the set of all positive imaginary roots by Δ_+^{im}. By definition, Δ decomposes as a disjoint union like $\Delta = \Delta^{\text{re}} \sqcup \Delta^{\text{im}}$.

2.2 Classification of generalized Cartan matrices

A generalized Cartan matrix (GCM) can be divided into three categories, each corresponding to a unique class of Kac-Moody algebra. The following propositions describe the three classes.

Proposition 2.2.1. *A Kac-Moody algebra is 'finite' iff all the principal minors of the corresponding GCM are positive.*

This is equivalent to the following:

F. 1 $det(A) \neq 0$;
F. 2 there exists a column vector $u \in \mathbb{R}^n > 0$ with $Au > 0$;
F. 3 $Au \geq 0$ implies $u > 0$ or $u = 0$.

Properties (F.1-F.3) imply that the associate algebra does not contain any imaginary roots.

Proposition 2.2.2. *A Kac-Moody algebra is 'affine' iff its GCM A satisfies,* $det(A) = 0$ *and all the proper minors of A are positive.*

This is equivalent to the following:

A. 1 $corank\, A = 1$;
A. 2 there exists a column vector $u > 0$ with $Au = 0$;
A. 3 $Au \geq 0$ implies $Au = 0$.

Properties (A.1-A.3) imply that the associate algebra contain imaginary roots.

Proposition 2.2.3. *A Kac-Moody algebra is called 'indefinite' iff its GCM A is indefinite type, i.e. if it satisfies* $det(A) < 0$ *and all the proper principal minors are negative.*

Equivalently,

I. 1 there exists $u > 0$ such that $Au < 0$;
I. 2 $Au \geq 0$ and $u \geq 0$ imply $u = 0$.

In the Propositions 2.2.1-2.2.3, u is assumed to be a column vector and lie in $\mathbb{R}^n$. To illustrate the differences of the three classes, consider the simplest non-trivial GCM, 2×2-dimensional

$$A = \begin{pmatrix} 2 & -r \\ -s & 2 \end{pmatrix}$$

where r and s are positive integers. Now classify the Kac-Moody algebra in each class associated with specific values for r and s.

1. Finite algebra: For this $det(A) > 0$ so $rs < 4$. There are three possibilities for non-equivalent algebras.
   a. $r = 1, s = 1$ corresponding to A_2;
   b. $r = 1, s = 2$ corresponding to B_2;
   c. $r = 1, s = 3$ corresponding to G_2.
2. Affine algebra: For this $det(A) = 0$ so $rs = 4$. There are two possibilities for non-equivalent algebras.
   a. $r = 1, s = 4$ corresponding to $A_2^{(2)}$;
   b. $r = 2, s = 2$ corresponding to $A_1^{(1)}$.
3. Indefinite algebra: For this $det(A) < 0$ so $rs > 4$. There are infinite number of choices resulting non-equivalent algebras.

Further, a GCM A is an indecomposable matrix, i.e. A cannot be written in the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ after reordering of indices. An indecomposable GCM can be either any of the the above type. A Kac-Moody Lie algebra is said to be any of finite, affine or indefinite Kac-Moody Lie algebra depending on whether the indecomposable GCM A is finite type, affine type or indefinite type respectively.

Given a GCM, one can associate a graph $S(A)$ with it, which is called Dynkin diagram. The vertices of $S(A)$ are labelled as $1, 2, \ldots, n$, where A is an GCM and suppose $i,\ j$ are distinct vertices. Now how the vertices are joined in $S(A)$, (depends on the pair (a_{ij}, a_{ji})), are described below:

1. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ number of lines. These lines are equipped with an arrow directing towards i if $|a_{ij}| > 1$.
2. If $a_{ij}a_{ji} > 4$, the vertices i and j are connected by boldface line equipped with an ordered pair of integers $|a_{ij}|$ and $|a_{ji}|$.

The relation between Cartan matrix, Dynkin diagram and an algebra is one-to-one. Clearly, A is indecomposable iff $S(A)$ is a connected graph. The Dynkin diagram corresponding to GCM is finite type if all its principal minors are positive definite, it is affine type if all its proper principal minors are positive definite and $detA = 0$ and otherwise indefinite. An indefinite type Dynkin diagram $S(A)$ is known as hyperbolic type if every proper connected subdiagrams of $S(A)$ are either of finite type or affine type, then the associated Kac-Moody algebra is called hyperbolic Kac-Moody algebra. In particular, $S(A)$ is of hyperbolic type if deletion of any of the vertex results in a diagram each of whose connected components are either finite or affine type.

The hyperbolic Kac-Moody algebras exist only in rank $2-10$ and have already been classified. The rank 2 hyperbolic Kac-Moody algebras are infinite in number. But the number of hyperbolic Kac-Moody algebras of rank > 2 are finite in number with maximum possible rank being 10. It has also been shown the rank 10 Kac-Moody algebra E_{10} contains every simply laced hyperbolic Kac-Moody subalgebra [FN04] as a Lie subalgebra [Vis08].

2.2.1 Root systems of finite dimensional semisimple Lie algebras(FSLA)

There is always a faithful representation for a semisimple Lie algebra $\mathfrak{g}$ is known as the adjoint representation

$$\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g}), \quad i.e., \quad \mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$$

and is defined as $\mathrm{ad}_x\,(y) = [x,y]$. Suppose $\mathfrak{h}$ is maximal abelian subalgebra known as Cartan subalgebra of $\mathfrak{g}$ w.r.t. which we have following root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \mathrm{ad}_h\, x = \alpha(h)x;\ \forall h \in \mathfrak{h}\}$. The nonzero $\alpha \in \mathfrak{h}^*$ for which $\mathfrak{g}_\alpha$ is nonzero is called a root and $\mathfrak{g}_\alpha$ is the corresponding root space. Δ is the set of all roots, known as root system of $\mathfrak{g}$. The following theorem will give clear significance of the root system (encoded in the Cartan matrix) associated to the Cartan decomposition of $\mathfrak{g}$.

Theorem 2.2.4. *To each root system (reduced) say R there exists a semisimple Lie algebra whose root system is isomorphic to R and two semisimple Lie algebras corresponding to isomorphic root systems are isomorphic.*

In Table 2.1 and 2.2, we have given the irreducible reduced root systems of complex semisimple Lie algebras: $A_l = \mathfrak{sl}(l+1,\mathbb{C})$ for $l \geq 1$, $B_l = \mathfrak{so}(2l+1,\mathbb{C})$ for $l \geq 2$, $C_l = \mathfrak{sp}(2l,\mathbb{C})$ for $l \geq 3$ and $D_l = \mathfrak{so}(2l,\mathbb{C})$ for $l \geq 4$, which are known as classical Lie algebras. Besides them there are other algebras E_6, E_7, E_8, G_2 and F_4 called exceptional Lie algebras. $V = \{v \in \mathbb{R}^{l+1} :< v, e_1 + \cdots + e_{l+1} >= 0\}$ is the under lying vector space for A_l and for rest algebras $V = \mathbb{R}^l$. The root system Δ is a subspace of some $\mathbb{R}^k = \sum_{i=1}^k a_i e_i$. Here $\{e_i\}$ is the standard orthonormal basis and a_i's are real. Δ_+ is the positive root system and Π is simple root system.

2.2.2 Root systems of affine untwisted Kac-Moody algebras

In this subsection, we give a more algebraic presentation of $\hat{\mathfrak{g}}$ which allows to have a unified point of view on finite dimensional semisimple Lie algebras and affine Kac-Moody algebras. This is an indication that affine Kac-Moody algebras are the natural generalizations of finite dimensional semisimple Lie algebras and so it is also a motivation for the definition of affine Kac-Moody algebras.

The untwisted affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ can be constructed from a finite di-

$\mathfrak{g}$	Δ	Δ^+	Π	Largest Root
A_l	$\{e_i - e_j \mid i \neq j\}$	$\{e_i - e_j \mid i < j\}$	$\{e_1 - e_2, \ldots, e_l - l + 1\}$	$e_1 - e_{l+1}$
B_l	$\{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\}$	$\{e_i \pm e_j \mid i < j\} \cup \{e_i\}$	$\{e_1 - e_2, \ldots, e_{l-1} - e_l, e_l\}$	$e_1 + e_2$
C_l	$\{\pm e_i \pm e_j \mid i < j\} \cup \{\pm 2e_i\}$	$\{e_i \pm e_j \mid i < j\} \cup \{2e_i\}$	$\{e_1 - e_2, \ldots, e_{l-1} - e_l, 2e_l\}$	$2e_1$
D_l	$\{\pm e_i \pm e_j \mid i < j\}$	$\{e_i \pm e_j \mid i < j\}$	$\{e_1 - e_2, \ldots, e_{l-1} - e_l, e_{l-1} + e_l\}$	$e_1 + e_2$

Table 2.1: Root systems of finite dimensional classical Lie algebras

$\mathfrak{g}$	Δ	Δ^+	Π	Largest Root
E_6	$\{\pm e_i \pm e_j \mid i < j \leq 5\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)} e_i \in V \mid \sum_{i=1}^{8} n(i) even\}$	$\{e_i \pm e_j \mid i > j\} \cup \{\frac{1}{2}(e_8 - e_6 - e_5 + \sum_{i=1}^{5}(-1)^{n(i)} e_i) \mid \sum_{i=1}^{5} n(i) even\}$	$\{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_4 - e_3, e_5 - e_4\}$	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$
E_7	$\{\pm e_i \pm e_j \mid i < j \leq 6\} \cup \{\pm(e_7 - e_8)\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)} e_i \in V \mid \sum_{i=1}^{8} n(i) even\}$	$\{e_i \pm e_j \mid i > j\} \cup \{e_8 - e_7\} \cup \{\frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{n(i)} e_i) \mid \sum_{i=1}^{6} n(i) odd\}$	$\{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5\}$	$e_8 - e_7$
E_8	$\{\pm e_i \pm e_j \mid i < j\} \cup \{\sum_{i=1}^{8}(-1)^{n(i)} e_i \mid \sum_{i=1}^{8} n(i) even\}$	$\{e_i \pm e_j \mid i > j\} \cup \{\frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{n(i)} e_i) \mid \sum_{i=1}^{7} n(i) even\}$	$\{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5, e_7 - e_6\}$	$e_7 + e_8$
F_4	$\{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\} \cup \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$	$\{e_i \pm e_j \mid i < j\} \cup \{e_i\} \cup \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_4)\}$	$\{\frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_4, e_3 - e_4, e_2 - e_3\}$	$e_1 + e_2$
G_2	$\{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\} \cup \{\pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$	$\{e_1 - e_2, e_3 - e_2, e_3 - e_1, -2e_1 + e_2 + e_3, e_1 - 2e_2 + e_3, 2e_3 - e_1 - e_2\}$	$\{e_1 - e_2, -2e_1 + e_2 + e_3\}$	$2e_3 - e_1 - e_2$

Table 2.2: Root systems of finite dimensional exceptional Lie algebras

mensional complex Lie algebra $\mathfrak{g}$. First we construct a loop algebra $\mathfrak{g}_{loop}$ which is the space of analytic single valued mappings from a circle S^1 to $\mathfrak{g}$. Then $\hat{\mathfrak{g}}$ is obtained by taking a non-trivial central extension of the loop algebra and further by adding a derivation term, which we discuss in next chapter with more detail. Here we give the general idea of the construction of the Chevalley generators of $\hat{\mathfrak{g}}$. Let $e_i, f_i, h_i (1 \leq i \leq n)$ be Chevalley generators of finite dimensional complex Lie algebra $\mathfrak{g}$. For $i = 1, \dots, n$ we set

$$\hat{e}_i = 1 \otimes e_i, \hat{f}_i = 1 \otimes f_i. \tag{2.2.1}$$

We also have the following decomposition for $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x \ \ \forall h \in \mathfrak{h}\}$ and $\Delta = \{\alpha \in \mathfrak{h}^* \backslash \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$ is the root system. Here we have $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$. Also, there is a $\theta \in \Delta$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for $i = 1, \dots, n$. Such θ is called the longest root of $\mathfrak{g}$.

Let ω be the linear involution of $\mathfrak{g}$ defined by

$$\omega(e_i) = -f_i, \ \omega(f_i) = -e_i, \ \omega(h_i) = -h_i. \tag{2.2.2}$$

Let us choose $f_0 \in \mathfrak{g}_\theta$ such that $(f_0, \omega(f_0)) = -\frac{2h^\vee}{(\theta,\theta)}$ and set $e_0 = -\omega(f_0) \in \mathfrak{g}_{-\theta}$. Now we have

$$[e_0, f_0] = (e_0, f_0)\nu^{-1}(\theta) = -\frac{2\theta}{(\theta,\theta)} = -\theta^\vee$$

where $(.,.)$ is the non-degenerate symmetric bilinear form on $\hat{\mathfrak{g}}$ and $\nu : \mathfrak{h} \to \mathfrak{h}^*$ is the isomorphism given by $< \nu(h), h_1 >= (h, h_1)$ (by using [Kac90], Theorem 2.2e). The linear functional $\lambda \in \mathfrak{h}^*$ can be extended to a linear functional on $\hat{\mathfrak{h}}$ by setting $< \lambda, c >= 0, < \lambda, d >= 0$ so that $\mathfrak{h}$ is identified as a subspace of $\hat{\mathfrak{h}}$. Let us consider the linear functional δ on $\mathfrak{h}$ is defined as $\delta|_{\mathfrak{h}+\mathbb{C}c} = 0$ and $< \delta, d >= 1$. Now set

$$\hat{e}_{n+1} = te_0, \ \hat{f}_{n+1} = t^{-1} f_0,$$

and

$$\hat{e}_i = e_i, \ \hat{f}_i = f_i \text{ for } \ i = 1, \dots, n.$$

The root system and the root space decomposition of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$ is

$$\hat{\Delta} = \{n\delta + \alpha : n \in \mathbb{Z}, \ \alpha \in \Delta\} \cup \{n\delta : n \in \mathbb{Z} \setminus \{0\}\}$$

and

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \hat{\Delta}} \hat{\mathfrak{g}}_\alpha$$

where $\hat{\mathfrak{g}}_{n\delta+\alpha} = t^n \mathfrak{g}_\alpha$ and $\hat{\mathfrak{g}}_{n\delta} = t^n \hat{\mathfrak{h}}$ respectively. If n is rank of $\mathfrak{g}$ then $\dim \hat{\mathfrak{g}}_{n\delta+\alpha} = 1$ and $\dim \hat{\mathfrak{g}}_{n\delta} = n$. By setting

$$\hat{\Pi} = \{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1} = \delta - \theta\}$$

and

$$\hat{\Pi}^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee, \alpha_{n+1}^\vee = \frac{2}{(\theta,\theta)} c - \theta^\vee\},$$

we get

$$A = (< \alpha_i^\vee, \alpha_j >)_{i,j=1}^{n+1}$$

and $(\hat{\mathfrak{h}}, \Pi, \Pi^\vee)$ is a realization of the affine matrix A. The simple roots of the untwisted affine Kac-Moody algebras are the simple roots of the Lie algebra $\mathfrak{g}$ completed by the root $(\delta - \theta)$, where θ is the largest root of $\mathfrak{g}$. The roots of the untwisted algebra are then

$$\hat{\Delta} = \{\alpha + n\delta : n \in \mathbb{Z}\} \cup \{n\delta : n \in \mathbb{Z}\backslash\{0\}\} \tag{2.2.3}$$

where α is the root of $\mathfrak{g}$. The roots $\alpha + n\delta$ have positive norm and are called real roots. They are non-degenerate. The roots $n\delta$ are degenerate and have zero norm. They are called imaginary roots. Root systems for all classical untwisted affine Kac-Moody algebras are given in the Table 2.3. Here we have deliberately omitted writing down root systems associated with exceptional untwisted Kac-Moody algebras, as we don't require these for our further studies.

$\hat{\mathfrak{g}}$	$\hat{\Delta}$	$\hat{\Delta}^+_{re}$	$\hat{\Pi}$
$A_l^{(1)}$	$\{\mathbb{Z}\delta \pm (e_i - e_j) \vert 1 \le i,j \le l+1, i \ne j\} \cup \{n\delta \vert n \in \mathbb{Z}\backslash\{0\}\}$	$\{n\delta \pm (e_i - e_j), (e_i - e_j) \mid i < j, n > 0\}$	$\{(e_i - e_{i+1}) + n\delta, m\delta \mid 1 \le i \le l, n \ge 0, m > 0\}$
$B_l^{(1)} (l \ge 2)$	$\{\mathbb{Z}\delta \pm e_i, \mathbb{Z}\delta \pm (e_i \pm e_j) \vert i \le j\} \cup \{n\delta \vert n \in \mathbb{Z}\backslash\{0\}\}$	$\{e_i \pm e_j, n\delta \pm (e_i \pm e_j), e_i, e_i + n\delta \mid 1 \le i < j \le l, n > 0\}$	$(e_i \pm e_{i+1}) + n\delta, \pm e_i + n\delta, m\delta \mid 1 \le i \le l, n \ge 0, m > 0\}$
$C_l^{(1)} (l \ge 2)$	$\{\mathbb{Z}\delta \pm 2e_i \mid 1 \le i \le l\} \cup \{\mathbb{Z}\delta \pm (e_i \pm e_j), j \ne 1, i < j\} \cup \{n\delta \mid n \in \mathbb{Z}\backslash\{0\}\}$	$\{e_i \pm e_j, n\delta \pm (e_i \pm e_j), 2e_i, \pm 2e_i + n\delta, \mid 1 \le i, j \ne 1, i < j, n > 0\}$	$\{e_i \pm e_{i+1} + n\delta, 2e_i + n\delta, m\delta \mid 1 \le i \le l, n \ge 0, m > 0\}$
$D_l^{(1)} (l \ge 4)$	$\{\mathbb{Z}\delta \pm (e_i \pm e_j) \vert 1 \le i < j \le l, \} \cup \{n\delta \vert n \in \mathbb{Z}\backslash\{0\}\}$	$\{e_i \pm e_j, n\delta \pm (e_i \pm e_j) \mid i < j, 1 \le i, j \le l, n > 0\}$	$\{e_i \pm e_{i+1} + n\delta, e_{l-1} + e_l + n\delta, m\delta \mid 1 \le i \le l, n \ge 0, m > 0\}$

Table 2.3: Root systems for classical untwisted affine Kac-Moody algebras

2.2.3 Root systems of affine twisted Kac-Moody algebras

We can similarly construct the twisted Kac-Moody affine algebras from the finite dimensional Lie algebra $\mathfrak{g}$. But in this case the analytic maps are not single valued. Thus, one should rather consider maps towards N-fold covering of the circle. The twisted algebras are associated to the outer automorphisms of the simple Lie algebras. Let σ be a finite order automorphism of $\mathfrak{g}$ preserving the Killing form. Fix a positive integer N such that $\sigma^N = id$ and set $\varepsilon = e^{2\pi i/N}$. Let $\mathfrak{g} = \bigoplus_s \mathfrak{g}_s$ be the corresponding $\mathbb{Z}/N\mathbb{Z}$-gradation, where $\mathfrak{g}_s = \{x \in \mathfrak{g} : \sigma(x) = \varepsilon^s x\}$. We extend σ to an automorphism of $\hat{\mathfrak{g}}$, preserving the Killing form and $\mathfrak{g}_{\text{loop}}$ by

$$c \mapsto c, \;\; d \mapsto d$$
$$t^k \otimes Y \mapsto (\varepsilon^{-1}t)^k \otimes \sigma(Y).$$

Let $\hat{\mathfrak{g}}(\sigma,N)$ and $\mathfrak{g}_{\text{loop}}(\sigma,N)$ denote the subalgebra of $\hat{\mathfrak{g}}$ and $\mathfrak{g}_{\text{loop}}$, respectively are the fixed point set of this automorphism. Therefore,

$$\mathfrak{g}_{\text{loop}}(\sigma,N) = \oplus_{s\in\mathbb{Z}}(t^s \otimes \mathfrak{g}_{s \bmod N})$$
$$\hat{\mathfrak{g}}(\sigma,N) = \mathfrak{g}_{\text{loop}}(\sigma,N) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The Lie algebra thus obtained is denoted by $\hat{\mathfrak{g}}(\sigma,N)$ and it is called a twisted affine Kac-Moody algebra. The inner automorphisms generate isomorphic algebras. Therefore, the twisted algebras are only related to the conjugacy classes of outer automorphisms. These classes are isomorphic to symmetries of the Dynkin diagram of $\mathfrak{g}$, which exist only in the cases A_l, D_l and E_6.

Let $\Delta(\mathfrak{g}_0)$ be the root system of $\mathfrak{g}_0$ and $\Delta(\mathfrak{g}_1)$ the non-zero weights of the $\mathfrak{g}_0$-representation $\mathfrak{g}_1$. The real roots of the twice twisted algebra, (i.e. algebra corresponds to outer automorphism twisted twice) are

$$\Delta^{\text{re}} = \{\Delta(\mathfrak{g}_0) + n\delta | n \in \mathbb{Z}\} \cup \{\Delta(\mathfrak{g}_1) + (n + \frac{1}{2})\delta | n \in \mathbb{Z}\}. \tag{2.2.4}$$

They are non-degenerate. The imaginary roots are

$$\Delta^{\text{im}} = \{n\delta/2 | n \in \mathbb{Z}\}. \tag{2.2.5}$$

Their degeneracy is rank($\mathfrak{g}_0$) if n is even and is rank($\mathfrak{g} - \mathfrak{g}_0$) if n is odd. Also there is only one twisted Kac-Moody algebra whose construction corresponds to an outer automorphism of order 3 (of D_4). The algebra is $D_4^{(3)}$ whose root system is given in Table 2.4. For $A_{2l-1}^{(2)}$ we denote $\eta_i = \sqrt{(e_i - e_{2l+1-i})}$ where $1 \le i \le l$ in Table 2.4. For $E_6^{(2)}$ we denote $\eta_1 = \frac{1}{2}(e_5 - e_6 - e_7 - e_8), \eta_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \eta_3 = \frac{1}{2}(-e_1 - e_2 + e_3 + e_4)$ and $\eta_4 = \frac{1}{2}(-e_1 + e_2 - e_3 + e_4)$. Again in case of $D_4^{(3)}$ we choose $\eta_1 = -e_4, \eta_4 = e_1, \eta_i = -e_i$ with $i \neq 1,4$.

2.3 Real forms, involutions and Vogan diagrams associated with FSLA

2.3.1 Real forms

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and we denote it by $\mathfrak{g}^{\mathbb{R}}$ when viewed as a real Lie algebra. A real Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}^{\mathbb{R}}$ is a real form of $\mathfrak{g}$ if $\mathfrak{g}$ is the complexifica-

$\mathfrak{g}$	Δ	Δ_{re}^{+}	Π
$A_{2l}^{(2)}$	$\{\mathbb{Z}\delta \pm (e_i \pm e_j), \mathbb{Z}\delta \pm e_i \mid 1 \leq i, j \leq l+1, i \neq j\} \cup \{n\delta \mid n \in \mathbb{Z}\backslash\{0\}\}$	$\{e_i, n\delta \pm e_i, e_i \pm e_j, n\delta(e_i \pm e_j), 2e_i, (2n+1)\delta \pm 2e_i \mid i < j, n > 0\}$	$\{e_i, n\delta \pm e_i, e_i \pm e_j, n\delta(e_i \pm e_j), 2e_i, (2n+1)\delta \pm 2e_i, m\delta \mid i < j, n > 0, m > 0\}$
$A_{2l-1}^{(2)}(l \geq 2)$	$\{\mathbb{Z}\delta \pm \frac{2\eta_i}{\sqrt{2}}, \frac{1}{2}n\delta + \sqrt{\frac{1}{2}}(\pm\eta_i \pm \eta_j) \mid 1 \leq i \neq j \leq l\} \cup \{n\delta \mid n \in \mathbb{Z}\backslash\{0\}\}$	$\{n\delta \pm \frac{2\eta_i}{\sqrt{2}}, \frac{1}{2}n\delta + \sqrt{\frac{1}{2}}(\pm\eta_i \pm \eta_j) \mid n > 0, 1 \leq i \neq j \leq l\}$	$\{n\delta \pm \frac{2\eta_i}{\sqrt{2}}, \frac{1}{2}n\delta + \sqrt{\frac{1}{2}}(\pm\eta_i \pm \eta_j), \frac{1}{2}m\delta \mid n > 0, m > 0, 1 \leq i \neq j \leq l\}$
$D_l^{(2)}(l \geq 4)$	$\{\mathbb{Z}\delta \pm (e_i \pm e_j), \pm e_i + \frac{1}{2}n\delta \mid 2 \leq i \leq j\} \cup \{\frac{1}{2}n\delta \mid n \in \mathbb{Z}\backslash\{0\}\}$	$\{\pm(e_i \pm e_j) + n\delta \mid i < j, 1 \leq i, j \leq l\}$	$\{e_i - e_{i+1} + n\delta, e_{l-1} + e_l + n\delta, m\delta \mid 1 \leq i \leq l, n \geq 0, m > 0\}$
$E_6^{(2)}$	$\{\pm\eta_i \pm \eta_j + n\delta, \pm\eta_i n\frac{\delta}{2}, \frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) + \frac{1}{2}n\delta, \frac{1}{2}m\delta \mid 1 \leq i \neq j, n \in \mathbb{Z}, m \in \mathbb{Z} \setminus 0\}$	$\{\eta_i \pm \eta_j + n\delta, \pm\eta_i + n\frac{\delta}{2}, \frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) + \frac{1}{2}n\delta \mid 1 \leq i \neq j, n > 0\}$	$\{\pm\eta_i \pm \eta_j + n\delta, \pm\eta_i + n\frac{\delta}{2}, \frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) + \frac{1}{2}n\delta, \frac{1}{2}m\delta \mid 1 \leq i \neq j, n > 0, m > 0\}$
$D_4^{(3)}$	$\{\pm(\eta_i - \eta_j) + n\delta, \frac{1}{3}(2\eta_i - \eta_j - \eta_k) + \frac{1}{3}n\delta, \frac{1}{3}m\delta\}$	$\{\pm(\eta_i - \eta_j) + n\delta, \pm\frac{1}{3}(2\eta_i - \eta_j - \eta_k) + \frac{1}{3}n\delta \mid n > 0, i \neq j\}$	$\{\pm(\eta_i - \eta_j) + n\delta, \pm\frac{1}{3}(2\eta_i - \eta_j - \eta_k) + \frac{1}{3}n\delta, \frac{1}{3}m\delta \mid n > 0, m > 0, i \neq j\}$

Table 2.4: Root systems for twisted affine Kac-Moody algebras

tion of $\mathfrak{g}_0$, i.e., if $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. Such a real form $\mathfrak{g}_0$ determines a mapping σ of $\mathfrak{g}$, namely $\sigma(x+iy) = x - iy$ for all $x, y \in \mathfrak{g}_0$ has properties:

- $\sigma[x,y] = [\sigma x, \sigma y]$,
- σ is semilinear, i.e., $\sigma(\mu x + \nu y) = \overline{\mu}\sigma(x) + \overline{\nu}\sigma(y)$,
- σ is an involution, i.e. $\sigma^2 = 1_{\mathfrak{g}}$,

$\mu, \nu \in \mathbb{C}$. The bijection $\sigma : \mathfrak{g} \to \mathfrak{g}$ is called a involution of $\mathfrak{g}$.

Conversely, any involution of $\mathfrak{g}$ determines uniquely a real subalgebra

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} : \sigma(x) = x\}$$

such that $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$. So a real form is the fixed point subalgebra of σ. Let Σ denote the set of all involution of $\mathfrak{g}$. Clearly we have

$$\Sigma \to \{\text{set of real forms of } \mathfrak{g}\}$$
$$\sigma \mapsto \mathfrak{g}^{\sigma},$$

bijection, i.e., there is a one-to-one correspondence between the involutions of complex Lie algebra $\mathfrak{g}$ and real forms of $\mathfrak{g}$. Infact the set Σ is non-empty as there are always a split real form and a compact real form to each $\mathfrak{g}$. The split real form is obtained by restricting the complex field to real and the compact real form $\mathfrak{u}$ is given explicitly by

$$\mathfrak{u} = \sum_{\alpha\in\triangle} \mathbb{R}i(h_\alpha) + \sum_{\alpha\in\triangle} \mathbb{R}(e_\alpha - f_\alpha) + \sum_{\alpha\in\triangle} \mathbb{R}i(e_\alpha + f_\alpha) \tag{2.3.1}$$

where $\{e_\alpha, f_\alpha, h_\alpha : \alpha \in \Delta\}$ are generators of $\mathfrak{g}$.

Definition 2.3.1. *Cartan-Killing form $B(x,y)$ of $\mathfrak{g}$ is a non-degenerate symmetric bilinear form defined as $B(x,y) = Tr(ad_x ad_y)$ for all $x,y \in \mathfrak{g}$.*

The Cartan-Killing form plays an important role in determining the real forms of a Lie algebra. In the following paragraph we point out some of its features with regard to real forms.

Let $B_{\mathfrak{g}_0}$ and $B_{\mathfrak{g}_0^{\mathbb{C}}}$ denote the Killing forms of the real Lie algebra $\mathfrak{g}_0$ and its complexification $\mathfrak{g}_0^{\mathbb{C}}$ respectively. Fix a basis for $\mathfrak{g}_0$, then it is also a basis for its complexification. So the Killing forms are related as $B_{\mathfrak{g}_0^{\mathbb{C}}}|_{\mathfrak{g}_0\times\mathfrak{g}_0} = B_{\mathfrak{g}_0}$ as $\mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y)$ is unaffected by complexifying. By Cartan's criterion for semisimplicity, a Lie algebra is semisimple iff Killing form is non-degenerate. So one can conclude $\mathfrak{g}_0^{\mathbb{C}}$ is semisimple iff $\mathfrak{g}_0$ is semisimple.

Let $\mathfrak{g}^{\mathbb{R}}$ denote the real Lie algebra obtained by restricting the scalars of complex Lie algebra $\mathfrak{g}$ and $B_{\mathfrak{g}^{\mathbb{R}}}$ be the corresponding Killing form. Let $\{v_j\}_{j\in I}$ be the basis for $\mathfrak{g}$, then $\{v_j, iv_j\}_{j\in I}$ is a basis for $\mathfrak{g}^{\mathbb{R}}$. Here the Killing forms are related by $B_{\mathfrak{g}^{\mathbb{R}}} = 2ReB_{\mathfrak{g}}$. Again by Cartan's criterion $\mathfrak{g}^{\mathbb{R}}$ is semisimple iff $\mathfrak{g}$ is semisimple. A real semisimple Lie algebra is said to be compact if its Killing form is negative definite.

2.3.2 Compact and split real form

For each root $\alpha \in \Delta$ it is possible to choose root vectors $x_\alpha \in \mathfrak{g}_\alpha$ such that for each pair of roots α, β we have

$$\begin{aligned}
[x_\alpha, x_\beta] &= N_{\alpha,\beta} x_{\alpha+\beta} && if\ \ \alpha+\beta \in \Delta \\
&= h_\alpha && if\ \ \alpha+\beta = 0 \\
&= 0 && \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Delta
\end{aligned}$$

and the constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ and $N_{\alpha,\beta}^2 = \frac{1}{2}q(1+p)|\alpha|^2$, where $\beta + n\alpha$ is the α string of β with $-p \le n \le q$. Clearly $N_{\alpha,\beta}^2$ is positive, so $N_{\alpha,\beta}$ is real. From these relations it follows $B(x_\alpha, x_\beta) = 1$ or 0 according as $\alpha+\beta$ is a root or not. Hence the elements $ih_\alpha, i(x_\alpha - x_\beta)$ and $(x_\alpha + x_\beta)$ span a compact real form of $\mathfrak{g}$. Again we can define another real form of $\mathfrak{g}$ by defining

$$\mathfrak{h}_0 = \{h \in \mathfrak{h} : \alpha(h) \in \mathbb{R}, \forall \alpha \in \Delta\}.$$

So the real form is given by

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus_{\alpha \in \Delta} \mathbb{R} x_\alpha. \tag{2.3.2}$$

Every real form of $\mathfrak{g}$ containing such an $\mathfrak{h}_0$ for some Cartan subalgebra $\mathfrak{h}$ is called split real form of $\mathfrak{g}$. So as mentioned earlier, split real form and compact real form do always exist for every complex semisimple Lie algebra $\mathfrak{g}$. In fact these two real forms are at opposite extremes and there are other real forms in between as well.

2.3.3 Cartan decomposition and Cartan involution

Definition 2.3.2. *An involution θ of a real semisimple Lie algebra is called a Cartan involution if the Killing form*

$$B_\theta(x,y) = -B(x, \theta y)$$

is positive definite.

If $\mathfrak{g}$ is a complex semisimple Lie algebra, $\mathfrak{u}$ is the compact real form and σ is a conjugation of $\mathfrak{g}$ w.r.t. $\mathfrak{u}$, then σ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$. To study arbitrary real form $\mathfrak{g}_0$ of a complex semisimple Lie algebra $\mathfrak{g}$, we want to align it with a compact real form, i.e., for $\mathfrak{g}_0$ we want to find $\mathfrak{u}$ such that $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{g}_0) \oplus (\mathfrak{u} \cap i\mathfrak{g}_0)$. This happens when the corresponding involutions of $\mathfrak{g}_0$ and $\mathfrak{u}$ commutes.

Theorem 2.3.3. *[Kna96a] Let $\mathfrak{g}_0$ be a real Lie algebra, θ a Cartan involution and σ be any involution of $\mathfrak{g}_0$. Then there exists a $\phi \in Int\mathfrak{g}_0$ such that $\phi\theta\phi^{-1}$ commutes with σ.*

Every real semisimple Lie algebra has a Cartan involution, also any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via inner automorphism of $\mathfrak{g}_0$. Again each compact real form has a determining associated conjugation. These conjugations are Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ and conjugate by member $\mathfrak{g}^{\mathbb{R}}$. Since $Int(\mathfrak{g}^{\mathbb{R}}) = Int(\mathfrak{g})$, hence any two compact real forms of $\mathfrak{g}$ are conjugate via $Int(\mathfrak{g})$.

Consider the real semisimple Lie algebra $\mathfrak{g}_0$. A vector space direct sum $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is called a Cartan decomposition if

1. $[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \;\; [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \;\; [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$
2. The Killing form $B_{\mathfrak{g}_0}$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$.

Here $\mathfrak{k}_0$ is a subalgebra but $\mathfrak{p}_0$ is a subspace of $\mathfrak{g}_0$. Also $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal w.r.t $B_{\mathfrak{g}_0}$ and B_θ.

The importance of Cartan decomposition is that it is unique upto conjugacy, i.e., if

$\mathfrak{g}_0 = \mathfrak{t}' \oplus \mathfrak{p}'$ is another Cartan decomposition of $\mathfrak{g}_0$ then there exists $\phi \in Int(\mathfrak{g}_o)$ such that $\mathfrak{t}' = \phi(\mathfrak{t})$ and $\mathfrak{p}' = \phi(\mathfrak{p})$.

There is a one-to-one correspondence between Cartan involutions and Cartan decompositions as follows. Given a Cartan involution θ, defines a eigenspace decomposition $\mathfrak{t}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$ into $+1$ and -1 eigen spaces and one can check this is a Cartan decomposition defined as above. Conversely starting from a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ we define a mapping θ which is $+1$ on $\mathfrak{t}_0$ and -1 on $\mathfrak{p}_0$. One can check θ respects the bracket and also B_θ is positive definite ($\mathfrak{t}$ and $\mathfrak{p}$ are orthogonal w.r.t. $B_{\mathfrak{g}_0}$ and B_θ, also $B_{\mathfrak{g}_0}$ is negative definite on $\mathfrak{t}_0$ and positive definite on $\mathfrak{p}_0$), hence θ is a Cartan involution.

For a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$, using the bilinearity of Killing form we see $\mathfrak{t}_0 + i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$. Conversely if $\mathfrak{l}_0$ and $\mathfrak{q}_0$ are the $+1$ and -1 eigenspaces of an involution σ, then σ is a Cartan involution only if the real form $\mathfrak{l}_0 + i\mathfrak{q}_0$ of $\mathfrak{g}_0^{\mathbb{C}}$ is compact. For $\mathfrak{g}$ a complex semisimple Lie algebra, $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{u}$ is a compact real form of $\mathfrak{g}_0$.

For a complex semisimple Lie algebra $\mathfrak{g}$, consider the compact real form $\mathfrak{u}$ and say the associated involution is $\sigma_{\mathfrak{u}}$. If σ is any other involution of $\mathfrak{g}$ then there exists an automorphism ϕ of $\mathfrak{g}$ such that $\sigma_{\mathfrak{u}}$ commutes with $\phi\sigma\phi^{-1}$. Hence to find all real forms of $\mathfrak{g}$, it is enough to find all involutions σ of $\mathfrak{g}$ that commutes with $\sigma_{\mathfrak{u}}$. Consider σ commutes with $\sigma_{\mathfrak{u}}$, we have $\sigma(\mathfrak{u}) = \mathfrak{u}$ and $\sigma(i\mathfrak{u}) = -i\sigma(\mathfrak{u})$. Let $\mathfrak{t}$ and $i\mathfrak{p}$ be $+1$ and -1 eigenspaces of σ in $\mathfrak{u}$ respectively, so that

$$\mathfrak{u} = \mathfrak{t} \oplus i\mathfrak{p}.$$

Again $+1$ and -1 eigenspaces of σ on $i\mathfrak{u}$ are $\mathfrak{p}$ and $i\mathfrak{t}$ so that

$$i\mathfrak{u} = \mathfrak{p} + i\mathfrak{t}.$$

Hence $\mathfrak{g}_0$ is the real form of $\mathfrak{g}$ determined by σ, we have

$$\mathfrak{g}_0 = \mathfrak{t} + \mathfrak{p}. \tag{2.3.3}$$

Here $\mathfrak{t} = \mathfrak{g}_0 \cap \mathfrak{u}$ is a subalgebra of $\mathfrak{g}_0$ and $\mathfrak{p} = \mathfrak{g}_0 \cap i\mathfrak{u}$ is a vector subspace such that

1. the Killing form is negative definite on $\mathfrak{t}$ and positive definite on $\mathfrak{p}$
2. the map $\sigma\sigma_{\mathfrak{u}} = \sigma_{\mathfrak{u}}\sigma = \theta : Y + Z \to Y - Z$ is an automorphism of $\mathfrak{g}_0$ for $Y \in \mathfrak{t}, Z \in \mathfrak{p}$.

The above decomposition of $\mathfrak{g}_0$ constructed from a compact real form $\mathfrak{u}$, of $\mathfrak{g}$ such that $\sigma_{\mathfrak{u}}$ commutes with σ, is a Cartan decomposition of real Lie algebra $\mathfrak{g}_0$, and θ is a Cartan involution of $\mathfrak{g}_0$.

From all of the above discussions it is clear that, classification of involutions lead to complete classification of all real forms of semisimple Lie algebra $\mathfrak{g}$. To be more precise for the classification of all real forms of a complex semisimple Lie algebra [Gil06] one has to find all automorphisms σ with $\sigma^2 = Id$ (i.e., all involutions) that commute with Cartan involutions. The only three different such σ are given below:

- σ is a complex conjugation
- $\sigma = \begin{pmatrix} +I_p & \cdots \\ \cdots & -I_q \end{pmatrix} = I_{p,q}$
- $\sigma = \begin{pmatrix} \cdots & +I_p \\ -I_p & \cdots \end{pmatrix} = J_{p,p}$

2.3.4 Vogan diagram

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, $\mathfrak{g}$ be the complexification as above. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Consider the Cartan involution θ and $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition. Let $\mathfrak{h}_0 = t_0 \oplus a_0$ be the θ-stable Cartan subalgebra of $\mathfrak{g}_0$ with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Dimension of t_0 is called the compact dimension and dimension of a_0 is the non-compact dimension $\mathfrak{h}_0$. So $\mathfrak{h}_0$ is said to be maximally compact if the dimension of t_0 is as large as possible and maximally non-compact if dimension of a_0 is as large as possible. Consider $\alpha \in \Delta$ is a root of $\mathfrak{g}$ (complexification of $\mathfrak{g}_0$) with respect to Cartan subalgebra $\mathfrak{h} (= \mathfrak{h}_0 \oplus i\mathfrak{h}_0)$. Then,

1. α is a real root if it is real valued on $\mathfrak{h}_0$ i.e. if it vanishes on t_0;
2. α is imaginary if it takes purely imaginary value on $\mathfrak{h}_0$ i.e. vanishes on a_0;
3. otherwise α is a complex root.

The θ-stable Cartan subalgebra $\mathfrak{h}_0$ is maximally noncompact if there are no noncompact imaginary roots and $\mathfrak{h}_0$ is maximally compact iff there are no real roots. Let $\mathfrak{h}_0$ be maximally compact with complexification being $\mathfrak{h}$. So Δ doesn't contain any real roots, i.e., no roots that vanish on $\mathfrak{t}$.

We may define a positive root system Δ_+ for Δ by choosing the first set of indices from $i\mathfrak{t}_0$ and next set from a basis of $\mathfrak{a}_0$. The roots in Δ_+ have atleast one non-vanishing component along $i\mathfrak{t}_0$, and the first non-zero one of these components is strictly positive. Since $\theta = +1$ on $\mathfrak{t}_0$ and since there are no real roots: $\theta(\Delta_+) = \Delta_+$. Thus θ permutes the simple roots, fixes imaginary roots and exchange in 2-tuples the complex roots: it constitutes an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$.

Definition 2.3.4. *By the Vogan diagram of the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta_+)$, we mean the Dynkin diagram of Δ_+ with additional information: the 2-element orbits under θ are so labeled and 1-element orbits painted and not painted according as the corresponding imaginary simple root is noncompact or compact respectively.*

Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be real semisimple Lie algebras. If we have two triples described as above and if they have same Vogan diagrams, then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic. Like Vogan diagram, we define an abstract Vogan diagram to be an abstract Dynkin diagram with two pieces of informations: one is an diagram automorphism of order 1 or 2 which can be indicated by labeling the 2-element orbits. The other is a subset of the 1-element orbits which is to be indicated by painting the vertices corresponding to the members of the subset. Every Vogan diagram is an abstract vogan diagrm. If an abstract Vogan diagram is given, then there exists a real semisimple Lie algebra $\mathfrak{g}_0$, a Cartan involution θ, maximally compact θ-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$, and a positive root system Δ_+ for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$ such that the given diagram is the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta_+)$.

Now for the complete classification of of all real semisimple real Lie algebras we have the following important result [Kna96b] which gives some allowable choices for Δ_+. Roughly the idea is that it is always possible to change Δ_+ so that atmost one simple imaginary root is painted.

Theorem 2.3.5 (Borel and de Siebenthal Theorem [Kna96a])**.** *Let $\mathfrak{g}_0$ be a non-complex simple real Lie algebra and Vogan diagram of $\mathfrak{g}_0$ be given corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta_+)$. Then there exists a simple system Π' for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, with corresponding positive root system Δ_+', such that $(\mathfrak{g}_0, \mathfrak{h}_0, {\Delta_+}')$ is a triple and there is atmost one painted simple root in its Vogan diagram. Furthermore suppose that the automorphism associated with the Vogan diagram is identity, that $\Pi' = \{\alpha_1, \ldots, \alpha_l\}$ and that $\{\omega_1, \ldots, \omega_l\}$ is the dual basis given by $(\omega_j, \alpha_k) = \delta_{jk}$. Then the single painted α_i may be chosen so that there is no i' with $(\omega_i - \omega_{i'}, \omega_{i'}) > 0$.*

The involutive automorphisms, real forms, Vogan diagrams play an important role in determining and classifying Riemmanian symmetric spaces. As our ultimate aim is to determine and classify the affine Kac-Moody symmetric spaces associated with classical untwisted affine Kac-Moody algebras, in this chapter we will provide a brief introductory note on symmetric spaces associated with FSLA. For details of the symmetric spaces one can refer [Hel79].

2.4 Symmetric spaces

A Riemannian manifold (M, d) is said to be a symmetric space (globally) if for every point $m \in M$ there exist an isometry $s_p : (M, d) \to (M, d)$ such that

1. $s_p(p) = p$
2. $\mathrm{d}s_p(p) = -\mathrm{Id}_{T_pM}$,

where T_pM is the tangent space of point p. This isometry is also known as involution at $p \in M$. We denote $I(M) = \{s_p \mid p \in M\}$ as the isometry group and $G \subset I(M)$ as the connected component of $I(M)$ containing identity element. The group G act on M transitively, i.e., for any two points r and s there exist an isometry which maps r onto s, means $s_p(r) = s$. With this action we want to define notation of isotropy group. Let us fix a point $p \in M$. Then, the closed subgroup

$$G_p = \{g \in G \mid g(p) = p\}$$

is called isotropy group and will be denoted as K. The differential at p of any $k \in K$ is an orthogonal transformation of T_pM. With this we view K as a closed subgroup of $O(T_pM)$ (the orthogonal group on T_pM). Thus, the embedding $K \hookrightarrow O(T_pM)$ is a representation of M known as isotropy representation.

Conversely, if M is any homogeneous space, i.e., its isometry group G acts transitively, then M is symmetric iff there exists a isometry s_p for some $p \in M$. Namely the symmetry at any other point $q = gp$ is the conjugate $s_q = gs_pg^{-1}$. Thus we have

Theorem 2.4.1. *A symmetric space M is precisely a homogeneous space with a symmetry s_p at some point $p \in M$.*

We may identify the homogeneous space M with the coset space G/K with G-equivariant diffeomorphism $gK \mapsto gp$.

We define an automorphism σ_p corresponding to isometry s_p as

$$\sigma_p : G \to G \text{ such that } \sigma_p(g) = s_p \circ g \circ s_p^{-1}. \tag{2.4.1}$$

Also $\sigma_p^2 = id$. Thus, σ_p is an involutive Lie group automorphism. Also it is important to note that

$$(\mathrm{d}\sigma_p)_e : \mathfrak{g} \to \mathfrak{g} \tag{2.4.2}$$

is Lie algebra automorphism.

Theorem 2.4.2.
1. *Let G be a connected Lie group. Let $\sigma : G \to G$ be an involutive automorphism (i.e. $\sigma^2 = id_G$) of the Lie group G. Define the fixed point set*

$$G^\sigma = \{g \in G : \sigma(g) = g\}$$

and K is the closed subgroup of the set G^σ. Then the symmetric space is the quotient space $M = G/K$ where the metric is induced from the given metric on G.

2. *Every symmetric space M arises in this way.*

As $\sigma \in \mathrm{Aut}(G), \sigma$ fixes the identity element. Hence, by differentiating at the identity, it induces an automorphism $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G, which is also an involution, i.e. $\sigma_*^2 = id$.

Theorem 2.4.3. *To any Lie algebra* $\mathfrak{g}$ *with Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *there exists unique simply connected symmetric space* $S = G/K$ *where* G *is the simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and* K *the connected subgroup with Lie algebra* $\mathfrak{k}$.

The Lie algebra $\mathfrak{g}$ of G is just the space of Killing vector fields of M. The Lie algebra $\mathfrak{k}$ of K is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is the natural complementary subspace of $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. It turns out that the characterization of symmetric space is the same as characterization of triple $(\mathfrak{g},\mathfrak{k},\mathfrak{p})$.

Definition 2.4.4. *Let* M *be a symmetric space and* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *as above. Then* M *is Euclidean type if* $[\mathfrak{p},\mathfrak{p}] = 0$. *It is compact type if* $\mathfrak{g}$ *is a semisimple Lie algebra and* M *is of nonnegative curvature.* M *is noncompact type if* $\mathfrak{g}$ *is semisimple and* M *is of nonpositive curvature.*

Now we give a brief review of practical way to construct symmetric spaces associated with a Lie algebra $\mathfrak{g}$. Algebraically the symmetric spaces can be obtained as follows: if $\mathfrak{g}$ is a compact simple real Lie algebra and σ is an involutive automorphism of $\mathfrak{g}$ we have, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\sigma(x) = x$ for $x \in \mathfrak{k}$ and $\sigma(x) = -x$ for $x \in \mathfrak{p}$. The following commutation relations also hold good i.e. $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. The subalgebra $\mathfrak{k}$ satisfying the above is called a symmetric subalgebra. Now multiply the elements in $\mathfrak{p}$ by i (the Weyl unitary trick) we construct a new non-compact algebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$. All non-compact real algebras are obtained in this way. Finally the coset spaces $exp(\mathfrak{p}) \simeq G/K$ and $exp(i\mathfrak{p}) \simeq G^*/K$ are globally Riemannian symmetric spaces.

2.4.1 Symmetric spaces associated with A_1

Consider the Lie algebra $A_1 = \mathfrak{sl}(2,\mathbb{C})$ which can be defined in a Chevelley basis as $[h,e] = 2e, [h,f] = -2f, [e,f] = h$. For this algebra the Chevelley generators are :

$$\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{2.4.3}$$

The Cartan antiinvolution which generates the compact real form is given by:

$$e \mapsto -f, \quad f \mapsto -e, \quad h \mapsto -h \tag{2.4.4}$$

Now the compact real form of A_1 is generated by $\{e - f, i(e+f), ih\}$. Explicitly we can write

$$\mathfrak{u} = a_1(ih) + a_2(e-f) + ia_3(e+f), \tag{2.4.5}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Using the matrix realization of e, f, h, we have

$$X = a_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + ia_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + i\,a_3 & -ia_1 \end{pmatrix} \quad (2.4.6)$$

The above matrix satisfies the properties of special unitary matrix (i.e. skew hermitian and trace is zero). So the compact real form of A_1 is $\mathfrak{su}(2)$. Now we want to find the other non-compact real forms of A_1. We have divided it into the following cases.

Case-I Let σ be an involutive automorphism on $\mathfrak{su}(2)$, i.e.

$$\sigma : \mathfrak{su}(2) \to \mathfrak{su}(2), \text{ such that } \sigma(X) = \bar{X},$$

where $X \in \mathfrak{su}(2)$. Under this automorphism

$$\begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + i\,a_3 & -ia_1 \end{pmatrix} \mapsto \begin{pmatrix} -ia_1 & a_2 - ia_3 \\ -a_2 - i\,a_3 & ia_1 \end{pmatrix} \quad (2.4.7)$$

which can be written as:

$$\begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix} + \begin{pmatrix} -ia_1 & -ia_3 \\ -ia_3 & ia_1 \end{pmatrix}. \quad (2.4.8)$$

Comparing with the Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ we have

$$\mathfrak{k} = \begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix} \in \mathfrak{so}(2), \quad \mathfrak{p} = \begin{pmatrix} -ia_1 & -ia_3 \\ -ia_3 & ia_1 \end{pmatrix}. \quad (2.4.9)$$

Thus by Weyl unitary trick

$$\mathfrak{k} + i\mathfrak{p} = \begin{pmatrix} a_1 & a_2 + a_3 \\ -a_2 + a_3 & -a_1 \end{pmatrix} \quad (2.4.10)$$

which is identified as $\mathfrak{sl}(2, \mathbb{R})$ and it is a non-compact real form of A_1.

Case-II Now defining another automorphism

$$\sigma(X) = I_{1,1} X I_{1,1} \quad (2.4.11)$$

where the matrix $I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We have

$$\sigma(X) = \begin{pmatrix} ia_1 & -a_2 - ia_3 \\ a_2 - i\,a_3 & -ia_1 \end{pmatrix}. \quad (2.4.12)$$

Comparing with the Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ we have

$$\mathfrak{k} = \begin{pmatrix} ia_1 & 0 \\ 0 & -ia_1 \end{pmatrix} \in so(2), \quad \mathfrak{p} = \begin{pmatrix} 0 & a_2 + ia_3 \\ -a_2 + ia_3 & 0 \end{pmatrix} \quad (2.4.13)$$

Similarly now

$$\mathfrak{t}+i\mathfrak{p}=\begin{pmatrix} ia_1 & ia_2-a_3 \\ -ia_2-a_3 & -ia_1 \end{pmatrix} \tag{2.4.14}$$

which is identified as $\mathfrak{su}(1,1)$ and it is known that $\mathfrak{sl}(2,\mathbb{R})\simeq\mathfrak{su}(1,1)$.

Hence the two symmetric spaces associated with A_1 are:

$$SU(2)/SO(2), \quad SL(2,R)/SO(2). \tag{2.4.15}$$

2.4.2 Symmetric spaces associated with A_2

The Chevalley generators of A_2 are

$$\left\{ e_1=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_1=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. e_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, h_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

Here the compact real form is generated by

$$\{(e_1-f_1),(e_2-f_2),(e_3-f_3),i(e_1+f_1),i(e_2+f_2),i(e_3+f_3),ih_1,ih_2\}$$

where

$$e_3=[e_1,e_2]=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_3=[f_2,f_1]=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The compact form is given by

$$X=\begin{pmatrix} ia_7 & a_1+ia_4 & a_3+ia_6 \\ -a_1+ia_4 & -ia_7+ia_8 & a_2+ia_5 \\ -a_3+ia_6 & -a_2+ia_5 & -ia_8 \end{pmatrix}$$

where $a_i\in\mathbb{R}$ for $i=1,\dots,8$. The trace of the above matrix is zero and satisfy the condition $X+X^*=0$ which is identified as $\mathfrak{su}(3)$. The other non-compact real forms are discussed in the following cases.

Case-I: Let $\sigma:\mathfrak{su}(3)\to\mathfrak{su}(3)$ such that $\sigma(X)=\bar{X}$, where $X\in\mathfrak{su}(3)$. Under this automorphism

$$\begin{pmatrix} ia_7 & a_1+ia_4 & a_3+ia_6 \\ -a_1+ia_4 & -ia_7+ia_8 & a_2+ia_5 \\ -a_3+ia_6 & -a_2+ia_5 & -ia_8 \end{pmatrix} \mapsto \begin{pmatrix} -ia_7 & a_1-ia_4 & a_3-ia_6 \\ -a_1-ia_4 & ia_7-ia_8 & a_2-ia_5 \\ -a_3-ia_6 & -a_2-ia_5 & ia_8 \end{pmatrix}$$

which can be written

$$\begin{pmatrix} 0 & a_1 & a_3 \\ -a_1 & 0 & a_2 \\ -a_3 & -a_2 & 0 \end{pmatrix} + \begin{pmatrix} -ia_7 & -ia_4 & -ia_6 \\ -ia_4 & -ia_8+ia_7 & -ia_5 \\ -ia_6 & -ia_5 & ia_8 \end{pmatrix}. \tag{2.4.16}$$

Comparing this with the Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ we have

$$\mathfrak{k} = \begin{pmatrix} 0 & a_1 & a_3 \\ -a_1 & 0 & a_2 \\ -a_3 & -a_2 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad \mathfrak{p} = \begin{pmatrix} -ia_7 & -ia_4 & -ia_6 \\ -ia_4 & -ia_8+ia_7 & -ia_5 \\ -ia_6 & -ia_5 & ia_8 \end{pmatrix}. \tag{2.4.17}$$

Thus by the Weyl unitary trick

$$\mathfrak{k} + i\mathfrak{p} = \begin{pmatrix} a_7 & a_1+a_4 & a_3+a_6 \\ -a_1+a_4 & -a_7+a_8 & a_2+a_5 \\ -a_3+a_6 & -a_2+a_5 & -a_8 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{R}). \tag{2.4.18}$$

Therefore the two symmetric spaces associated with A_2 are:

$$SL(3,R)/SO(3), \quad SU(3)/SO(3). \tag{2.4.19}$$

Case-II: Now considering another automorphism

$$\sigma(X) = I_{2,1} X I_{2,1} \tag{2.4.20}$$

where the matrix $I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The compact form of A_2

$$\begin{pmatrix} ia_7 & a_1+ia_4 & a_3+ia_6 \\ -a_1+ia_4 & -ia_7+ia_8 & a_2+ia_5 \\ -a_3+ia_6 & -a_2+ia_5 & -ia_8 \end{pmatrix}$$

can be written in this form

$$\begin{pmatrix} (A)_{2\times 2} & (B)_{2\times 1} \\ (-B^*)_{1\times 2} & (C)_{1\times 1} \end{pmatrix}. \tag{2.4.21}$$

Now applying the automorphism (2.4.20) on compact form we have

$$\begin{pmatrix} A & B \\ (-B^*) & C \end{pmatrix} \longrightarrow \begin{pmatrix} A & -B \\ B^* & C \end{pmatrix}. \tag{2.4.22}$$

Here, $\mathfrak{k} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} ia_7 & a_1 + ia_4 & 0 \\ -a_1 + ia_4 & -ia_7 + ia_8 & 0 \\ 0 & 0 & -ia_8 \end{pmatrix}$ and $\mathfrak{p} = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}$.

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} A - \frac{1}{2}Tr(A)I_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}Tr(A)I_2 & 0 \\ 0 & Tr(C) \end{pmatrix} \tag{2.4.23}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & C - Tr(C) \end{pmatrix} \tag{2.4.24}$$

shows that $\mathfrak{k}$ is isomorphic to $\mathfrak{su}(2) \times c_0 \times \mathfrak{su}(1)$ where c_0 is the center of $\mathfrak{k}$. Thus,

$$\mathfrak{k} + i\mathfrak{p} = \begin{pmatrix} ia_7 & a_1 + ia_4 & ia_3 - a_6 \\ -a_1 + ia_4 & -ia_7 + ia_8 & ia_2 - a_5 \\ -ia_3 - a_6 & -ia_2 - a_5 & -ia_8 \end{pmatrix} \in \mathfrak{su}(2,1). \tag{2.4.25}$$

Hence finally the two symmetric spaces associated with the algebra A_2 are given by:

$$SU(2,1)/S(U_2 \times U_1), \quad SU(2+1)/S(U_2 \times U_1). \tag{2.4.26}$$

The simply connected Riemannian symmetric spaces corresponding to the pair $(\mathfrak{u}, \sigma)$ and $\mathfrak{g}_0$ (for $\mathfrak{u}$ classical) [Hel79] as discussed above are given in the Table 2.5 below.

Complex Lie algebra	$\mathfrak{u}$	σ	$\mathfrak{t}_o$	compact symmetric space	Non-compact symmetric space
A_n	$\mathfrak{su}(n)$	$\sigma(X) = \overline{X}$	$\mathfrak{so}(n)$	$\frac{SU(n)}{SO(n)}, n > 1$	$\frac{SL(n,\mathbb{R})}{SO(n)}, n > 1$
	$\mathfrak{su}(2n)$	$\sigma(X) = J_n\overline{X}J_n^{-1}$	$\mathfrak{sp}(n)$	$\frac{SU(2n)}{Sp(n)}, n > 1$	$\frac{SU^*(2n)}{Sp(n)}, n > 1$
	$\mathfrak{su}(p+q)$	$\sigma(X) = I_{p,q}XI_{p,q}$	$\mathfrak{su}(p) \times c_0 \times \mathfrak{su}(q)$	$\frac{SU(p+q)}{S(U_p \times U_q)}, p \geq 1, q \geq 1, p \geq q$	$\frac{SU(p,q)}{S(U_p \times U_q)}, p \geq 1, q \geq 1, p \geq q$
B_n	$\mathfrak{so}(p+q)$	$\sigma(X) = I_{p,q}XI_{p,q},\ p \geq q$	$\mathfrak{so}(p) \times \mathfrak{so}(q)$	$\frac{SO(p+q)}{SO(p) \times SO(q)}, p > 1, q \geq 1, p+q \neq 4, p \geq q$	$\frac{SO(p,q)}{SO(p) \times SO(q)}, p > 1, q \geq 1, p+q \neq 4, p \geq q$
C_n	$\mathfrak{sp}(n)$	$\sigma(X) = \overline{X}$	$\mathfrak{u}(n)$	$\frac{Sp(n)}{U(n)}, n \geq 1$	$\frac{Sp(n.\mathbb{R})}{U(n)}, n \geq 1$
	$\mathfrak{sp}(p+q)$	$\sigma(X) = K_{p,q}XK_{p,q}$	$\mathfrak{sp}(p) \times \mathfrak{sp}(q)$	$\frac{Sp(p+q)}{Sp(p) \times sp(q)}, p \geq q \geq 1$	$\frac{Sp(p,q)}{Sp(p) \times Sp(q)}, p \geq 1, q \geq 1$
D_n	$\mathfrak{so}(2n)$	$\sigma(X) = J_nXJ_n^{-1}$	$\mathfrak{u}(n)$	$\frac{SO(2n)}{U(n)}, n > 2$	$\frac{SO^*(2n)}{Un}, n > 2$

Table 2.5: Riemannian symmetric spaces for classical Lie algebras

Chapter 3

Affine Kac-Moody symmetric spaces and classifications

In this chapter, we will start with describing about the central extensions of the Lie algebra and loop algebra which will help us to realize the untwisted affine Kac-Moody algebras [Kac90].

3.1 Realization of affine untwisted Kac-Moody algebra

3.1.1 Central extensions

We can construct the central extension of a Lie algebra $\mathfrak{g}$ in the following way: The extension of a Lie algebra by central elements gives another Lie algebra $\bar{\mathfrak{g}} = V \oplus \mathfrak{g}$, where V is a vector space. The Lie bracket of this new Lie algebra is given by:

$$[(v,x),(w,y)] := (\Omega(x,y),[v,w]) \tag{3.1.1}$$

where $\Omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Since this bracket must be antisymmetric, bilinear and satisfy the Jacobi identity, thus $\Omega(x,y)$ must be antisymmetric, bilinear and satisfy:

$$\Omega(x,[y,z]) + \Omega(y,[z,x]) + \Omega(z,[x,y]) = 0 \tag{3.1.2}$$

i.e., Ω is cocycle of $\mathfrak{g}$. The central extension is trivial precisely if Ω is a coboundary, i.e. when $\Omega(x,y)$ is a linear function of $[x,y]$. The inequivalent non-trivial central extensions are then described by the vector space of 2-cocycles modulo coboundaries, i.e. the (Lie algebra) cohomology $H^2(\mathfrak{g},\mathbb{C})$. An alternative way to relate $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ is is by requiring

$$0 \to \mathbb{C} \to \bar{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{3.1.3}$$

to be short exact sequence.

3.1.2 Loop algebra

The loop algebra associated to a Lie algebra $\mathfrak{g}$ is the space of analytic (single valued) mappings from the circle S^1 to $\mathfrak{g}$. Let $\{T^a : a = 1,\dots,d\}$ be a basis of the d-dimensional Lie algebra $\mathfrak{g}$ with the structure constants f_c^{ab} and t a complex coordinate on S^1, then a basis of the loop algebra $\mathfrak{g}_{\text{loop}}$ associated to $\mathfrak{g}$ is given by

$$B = \{T_n^a : a = 1,\dots,d; n \in \mathbb{Z}\} \tag{3.1.4}$$

where $T_n^a := T^a \otimes t^n = T^a \otimes e^{2\pi i t n}$. So we can write the loop algebra as

$$\mathfrak{g}_{\text{loop}} = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}, \tag{3.1.5}$$

where $\mathbb{C}[t,t^{-1}]$ denotes the algebra of Laurent polynomials in t (elements are of the form $\sum_{k\in\mathbb{Z}} c_k t^k$ where only a finite number of the complex parameters c_k are non zero). This space inherits a natural bracket operation from $\mathfrak{g}$, i.e.,

$$\begin{aligned}[T_m^a, T_n^b] &= [T^a \otimes t^m, T^b \otimes t^n] := [T^a, T^b] \otimes t^m t^n \\ &= \sum_{c=1}^{d} f_c^{ab} T^c \otimes t^{m+n} = \sum_{c=1}^{d} f_c^{ab} T_{m+n}^c\end{aligned}$$

$\forall T^a \in \mathfrak{g}; \forall t^m, t^n \in \mathbb{C}[t,t^{-1}]$. With this bracket $\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ becomes Lie algebra. But the loop algebras $\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ are not yet affine Lie algebras. The center of the loop algebra based on a simple Lie algebra is trivial. Thus, let us consider a central extension of a loop algebra of simple Lie algebras. Define a $\mathbb{C}$-valued 2-cocycle on the loop algebra, i.e.,

$$\psi(a,b) = \kappa(x,y)\varphi(P,Q) \tag{3.1.6}$$

where $a = P \otimes x, b = Q \otimes y, P, Q \in \mathbb{C}[t,t^{-1}], x, y \in \mathfrak{g}$ and

$$\varphi(P,Q) := \operatorname{Res} \frac{dP}{dt} Q, \tag{3.1.7}$$

$\kappa(x,y)$ is a non-degenerate, invariant, symmetric complex valued bilinear form. Also, the maps

$$\frac{d}{dt} : \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \to \mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}; \text{ and } \operatorname{Res} : \mathbb{C}[t,t^{-1}] \to \mathbb{C}$$

are defined by

$$\frac{d}{dt}(P \otimes x) = \frac{dP}{dt} x \text{ for } P \in \mathbb{C}[t,t^{-1}], x \in \mathfrak{g}; \text{ and } \operatorname{Res}(t^r) = \delta_{r,-1} \text{ for } r \in \mathbb{Z}.$$

Now we have the following proposition.

Proposition 3.1.1. *There exists a non-trivial central extensions* $\bar{\mathfrak{g}}$ *of* $\mathfrak{g}_{loop}$ *given as*

$$\bar{\mathfrak{g}} = \mathfrak{g}_{loop} \oplus \mathbb{C}c = (\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c \tag{3.1.8}$$

whose bracket is given by

$$[a+\lambda c, b+\mu c] = [a,b] + \psi(a,b)c \tag{3.1.9}$$

$a,b \in \mathfrak{g}_{loop}$; $\lambda, \mu \in \mathbb{C}$.

Now we can see that the extended loop algebra $\bar{\mathfrak{g}} = (\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c$ is infinite dimensional. Moreover, the algebra $\bar{\mathfrak{g}}$ is a derived algebra, i.e., $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = \bar{\mathfrak{g}}$. In other words, any element $z \in \bar{\mathfrak{g}}$ can be written as a Lie bracket $[x,y]$ of two elements. Therefore, the bilinear form in this algebra is degenerate. This can be remedied by adding an extra generator d, called a derivation, that cannot be written as the commutator of two elements.

Finally, we denote by $\hat{\mathfrak{g}}$ the Lie algebra, which is obtained by adjoining to $\bar{\mathfrak{g}}$ a derivation which acts on $\bar{\mathfrak{g}}$ as $t\frac{d}{dt}$ and kills c. Explicitly

$$\hat{\mathfrak{g}} = (\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with bracket defined by

$$[d, P(t) \otimes x] = t\frac{dP(t)}{dt} \otimes x, \text{ and } [d,c] = 0 \tag{3.1.10}$$

for $P(t) \in \mathbb{C}[t,t^{-1}]$ and $x \in \mathfrak{g}$. Now this $\hat{\mathfrak{g}}$ is the non-twisted affine Kac-Moody Lie algebra associated with the affine matrix A of type $X_n^{(1)}$.

Remark 3.1.2. *The map $\mathfrak{g} \to \hat{\mathfrak{g}}$ is a Lie algebra embedding. In particular, we consider $\mathfrak{g}$ as a subalgebra of $\hat{\mathfrak{g}}$. We define Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ as $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$.*

There is an abundance of literature available for affine Kac-Moody algebras [Kac68b, Kac90, Moo67], affine Kac-Moody groups, their involutive automorphisms, real forms, Vogan diagrams etc. [Bat00, Bat02, MR03, Kob86, Lev88, Cor92b, Cor92c]. In this section we shall briefly outline all the materials related to these algebras which are essential to our studies.

3.2 Automorphisms and real forms of non-twisted affine Kac-Moody algebras

Define a group G acting on the algebra $\hat{\mathfrak{g}}$ through adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\hat{\mathfrak{g}})$. It is generated by the subgroup U_α, for $\alpha \in \hat{\Delta}^{\mathrm{re}}$, which are isomorphic to the additive groups $\hat{\mathfrak{g}}_\alpha$ by an isomorphism exp such that $\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}$.

A maximal $\mathrm{ad}_{\hat{\mathfrak{g}}}$-diagonalizable subalgebra of $\hat{\mathfrak{g}}$ is called a Cartan subalgebra (CSA). Every CSA is $\mathrm{Ad}(G)$-conjugate to the standard Cartan subalgebra $\hat{\mathfrak{h}}$. A Borel subalgebra

(BSA) is maximal completely solvable subalgebra of $\hat{\mathfrak{g}}$. Consider $\hat{\mathfrak{b}}^+ = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha>0} \hat{\mathfrak{g}}_\alpha$ and $\hat{\mathfrak{b}}^- = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha<0} \hat{\mathfrak{g}}_\alpha$, , are respectively called the positive and negative standard BSA's. The subalgebras $\hat{\mathfrak{b}}^+$ and $\hat{\mathfrak{b}}^-$ are not conjugate by $\mathrm{Ad}(G)$. All the BSA's conjugate to $\hat{\mathfrak{b}}^+$ (respectively $\hat{\mathfrak{b}}^-$) are said to be positive (respectively negative). If $\hat{\mathfrak{g}}$ is indecomposable, all BSAs are either positive or negative [KP83].

An automorphism (linear or semi-linear) of $\hat{\mathfrak{g}}$ acts in a compatible manner to Ad on G and hence transforms two conjugate BSAs to two conjugate BSAs; it is said to be of first type (respectively second type) if it transforms a positive BSA to positive (respectively negative) BSA. If $\hat{\mathfrak{g}}$ is indecomposable, all automorphisms are either of first or second type.

3.2.1 Automorphism of $\hat{\mathfrak{g}}$

The group of automorphisms [Rou88] of $\hat{\mathfrak{g}}$ is given by

$$\mathrm{Aut}(\hat{\mathfrak{g}}) = [\{1, \omega\} \times \mathrm{Aut}(A) \ltimes Int(\hat{\mathfrak{g}})] \ltimes \mathrm{Tr} \tag{3.2.1}$$

where ω is the Cartan involution, $\mathrm{Aut}(A)$ is the group of permutations ρ of I such that $a_{\rho_i \rho_j} = a_{ij}$ for $i, j \in I$, $\mathrm{Int}(\hat{\mathfrak{g}})$ is the set of interior automorphisms of $\hat{\mathfrak{g}}$ and Tr= $\mathrm{Tr}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}', c)$ is the group of transvections of $\hat{\mathfrak{g}}$ [MR03].

Definition 3.2.1. *[BR89, p.190] Let $Aut_{\mathbb{R}}(\hat{\mathfrak{g}})$ denote the group of automorphisms of $\hat{\mathfrak{g}}$ that are either $\mathbb{C}$-linear or semilinear (i.e., $\phi(\lambda x) = \bar{\lambda}\phi(x), \lambda \in \mathbb{C}, x \in \hat{\mathfrak{g}}$). $Aut(\hat{\mathfrak{g}})$ is an index 2 normal subgroup of $Aut_{\mathbb{R}}(\hat{\mathfrak{g}})$.*

A semi-involution of $\hat{\mathfrak{g}}$ is a semi-linear automorphism of order 2. For all semi-involutions σ' we have a decomposition, $\mathrm{Aut}_{\mathbb{R}}(\hat{\mathfrak{g}}) = \{1, \sigma'\} \ltimes Aut(\hat{\mathfrak{g}})$. If σ' is a semi-involution of $\hat{\mathfrak{g}}$, the real Lie algebra $\hat{\mathfrak{g}}_{\mathbb{R}} = \hat{\mathfrak{g}}^{\sigma'}$ is a real form of $\hat{\mathfrak{g}}$, in the sense that there exists an isomorphism of the complex Lie algebras $\hat{\mathfrak{g}}_{\mathbb{R}} \otimes \mathbb{C}$ and $\hat{\mathfrak{g}}$ [BR89]. Further, σ' is the conjugation of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{g}}_{\mathbb{R}}$. Thus there exists a bijective correspondence between the semi-involutions and real forms. The standard normal (or split) real form of $\hat{\mathfrak{g}}$ is the real Lie algebra generated by $\hat{e}_i, \hat{f}_i, \hat{h}_i$ and d. The corresponding semi-involution σ'_n is called the normal semi-involution. Also, σ'_n commutes with the standard Cartan involution ω.

The standard Cartan semi-involution ω'_s of $\hat{\mathfrak{g}}$ is the unique semi-involution of $\hat{\mathfrak{g}}$ such that $\omega'_s(\hat{e}_i) = -\hat{f}_i$ and $\omega'_s(d) = -d$. Therefore, $\omega'_s = \sigma'_n\omega$. All conjugates of ω'_s are called Cartan semi-involutions or compact semi-involutions ω'; these are semi-involutions of the second type. The corresponding real forms are called the compact real forms. It is clear that, for all affine Kac-Moody Lie algebras, there exists, upto a conjugation, a unique compact real form.

3.2.2 Real form of $\hat{\mathfrak{g}}$

The real form corresponding to a semi-involution of first kind (respectively of second kind is said to be almost split (respectively almost compact) real form. Upto a conjugation, a classification of the almost split real forms is given in [BBMR95] and a classification of the almost compact real forms is given in [MR03].

3.2.3 Cartan subalgebra of a real form of $\hat{\mathfrak{g}}$

Let $\hat{\mathfrak{g}}_{\mathbb{R}}$ be real form of $\hat{\mathfrak{g}}$. A Lie subalgebra of $\hat{\mathfrak{h}}_0$ of $\hat{\mathfrak{g}}_{\mathbb{R}}$ is called the Cartan subalgebra of $\hat{\mathfrak{g}}_{\mathbb{R}}$ if the complexification of $\hat{\mathfrak{h}}_0 \otimes \mathbb{C}$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$.

3.2.4 Cartan involution

Let σ' be a semi-involution of $\hat{\mathfrak{g}}$ of second kind and let $\hat{\mathfrak{g}}_{\mathbb{R}} = \hat{\mathfrak{g}}^{\sigma'}$ be the corresponding almost compact real form. An involution ω' which commutes with σ' is called a Cartan semi-involution for σ' or $\hat{\mathfrak{g}}_{\mathbb{R}}$. The involution $\sigma = \sigma'\omega'$ is called a Cartan involution of σ' and also the restriction $\omega'_{\mathbb{R}}$ of σ to $\hat{\mathfrak{g}}_{\mathbb{R}}$ is called Cartan involution for $\hat{\mathfrak{g}}_{\mathbb{R}}$.

Let us take $\hat{\mathfrak{u}} = \hat{\mathfrak{g}}^{\omega'}$. The algebra of fixed points $\hat{\mathfrak{k}}_0 = \hat{\mathfrak{g}}^{\sigma}_{\mathbb{R}} = \hat{\mathfrak{g}}_{\mathbb{R}} \cap \hat{\mathfrak{u}} = \hat{\mathfrak{u}}^{\sigma}$is called a maximal compact subalgebra of $\hat{\mathfrak{g}}_{\mathbb{R}}$. Now we have the Cartan decomposition $\hat{\mathfrak{g}}_{\mathbb{R}} = \hat{\mathfrak{k}}_0 \oplus \hat{\mathfrak{p}_0}$ and $\hat{\mathfrak{u}} = \hat{\mathfrak{k}}_0 \oplus i\hat{\mathfrak{p}_0}$ where $\hat{\mathfrak{p}}_0$ is the eigenspace of $\omega'_{\mathbb{R}}$ for eigen value -1 [BR89, p.199-200]. Let t_0 be a maximal abelian subspace of $\mathfrak{t}_0$. Then $\mathfrak{h}_\circ = Z_{\mathfrak{g}_{\mathbb{R}}}(t_0)$ is a σ-stable Cartan subalgebra of the almost compact real form $\mathfrak{g}_{\mathbb{R}}$ of the form $\mathfrak{h}_\circ = t_0 \oplus a_0$ with $a_0 \subseteq \mathfrak{p}_\circ$ [BR89, Rou88].

A σ-stable Cartan subalgebra $\hat{\mathfrak{h}}_0 = t_0 \oplus a_0$ with $t_0 \subseteq \hat{\mathfrak{k}}_0$ and $a_0 \subseteq \hat{\mathfrak{p}}_0$ of an almost compact real form $\hat{\mathfrak{g}}_{\mathbb{R}}$ is maximally compact if the dimension of t_0 is as large as possible and it is maximally non-compact if the dimension of a_0 is as large as possible. A maximally compact Cartan subalgebra $\hat{\mathfrak{h}}_0$ of an almost compact real form $\hat{\mathfrak{g}}_{\mathbb{R}}$ has the property that all the roots are real on a_0 and imaginary on t_0. One can say that a root is real if it takes real value on $\hat{\mathfrak{h}}_0 = t_0 \oplus a_0$, i.e. vanishes on t_0. It is imaginary if it takes imaginary value on $\hat{\mathfrak{h}}_0$, i.e. vanishes on a_0 and complex otherwise.

3.2.5 Classification of real forms

There is a one–one correspondence between the conjugacy classes of (linear) involutions of the second kind of $\hat{\mathfrak{g}}$ and the conjugacy classes of almost split real forms of $\hat{\mathfrak{g}}$ under Aut($\hat{\mathfrak{g}}$) [BBMR95, Theorem 4.4].

Theorem 3.2.2 ([MR03]). *We consider*

1. *the semi-involutions σ' of $\hat{\mathfrak{g}}$, of the second kind;*
2. *the involutions θ, of $\hat{\mathfrak{g}}$ the first kind;*

3. *the relation $\sigma' \approx \theta$ if and only if*
   (a) $\omega' = \theta\sigma' = \sigma'\theta$ *is a Cartan semi-involution,*
   (b) θ *and* σ' *stabilize the same Cartan subalgebra* $\hat{\mathfrak{h}}$,
   (c) $\hat{\mathfrak{h}}$ *is contained in a minimal* σ'*-stable positive parabolic subalgebra.*

Then this relation induces a bijection between the conjugacy classes under Aut($\hat{\mathfrak{g}}$) of semi-involutions of the second kind and conjugacy classes of involutions of the first kind.

We see there is a bijection between the conjugacy classes of semi-involution of second kind and conjugacy classes of involution of first kind under Aut$(\hat{\mathfrak{g}})$. Thus one obtain under Aut$(\hat{\mathfrak{g}})$ a one-to-one correspondence between conjugacy classes of linear involutions of first kind (including identity) and the the conjugacy classes of almost compact real forms of $\hat{\mathfrak{g}}$. The compact real form is unique and corresponds to the identity.

Let $\hat{\mathfrak{g}}_{\mathbb{R}}$ be almost compact real form of $\hat{\mathfrak{g}}$ corresponding to the semi-involution of the second kind σ' of $\hat{\mathfrak{g}}$. Let σ be the Cartan involution of $\hat{\mathfrak{g}}_{\mathbb{R}}$ and let $\hat{\mathfrak{g}}_{\mathbb{R}} = \hat{\mathfrak{k}}_0 \oplus \hat{\mathfrak{p}}_0$ be the corresponding Cartan decomposition [Bir37, Jac79]. Let t_0 be maximal abelian subspace of $\hat{\mathfrak{k}}_0$. Then $\hat{\mathfrak{h}}_0 = Z_{\hat{\mathfrak{g}}_{\mathbb{R}}}(t_0)$ is a σ-stable Cartan subalgebra of $\hat{\mathfrak{g}}_{\mathbb{R}}$ of the form $\hat{\mathfrak{h}}_0 = t_0 \oplus a_0$ with $a_0 \subseteq \hat{\mathfrak{p}}_0$. This $\hat{\mathfrak{h}}_0$ is a maximally Cartan subalgebra of $\hat{\mathfrak{g}}_{\mathbb{R}}$ because t_0 is as large as possible.

For any root α, $\sigma(\alpha)$ is the root $\sigma\alpha(\hat{h}) = \alpha(\sigma^{-1}\hat{h})$. If α is imaginary then $\sigma(\alpha) = \alpha$ and α vanishes on a_0. Thus $\hat{\mathfrak{g}}_\alpha$ is σ-stable and we have $\hat{\mathfrak{g}}_\alpha = (\hat{\mathfrak{g}}_\alpha \cap \hat{\mathfrak{k}}) \oplus (\hat{\mathfrak{g}}_\alpha \cap \hat{\mathfrak{p}})$. Again $\dim(\hat{\mathfrak{g}}_\alpha) = 1$, so $\hat{\mathfrak{g}}_\alpha \subseteq \hat{\mathfrak{k}}$ or $\hat{\mathfrak{g}}_\alpha \subseteq \hat{\mathfrak{p}}$. An imaginary root α is called compact if $\hat{\mathfrak{g}}_\alpha \subseteq \hat{\mathfrak{k}}$ and is non-compact if $\hat{\mathfrak{g}}_\alpha \subseteq \hat{\mathfrak{p}}$. Let $\hat{\mathfrak{h}}_0$ be a σ-stable Cartan subalgebra of $\hat{\mathfrak{g}}_{\mathbb{R}}$. Then there are no real roots iff $\hat{\mathfrak{h}}_0$ is maximally compact [Bat00].

For classification of real forms of affine Kac-Moody algebra there are two main approach: one focuses the maximal non-compact Cartan subalgebra that leads to Satake diagrams [TP06]. The other one is with maximal compact Cartan subalgebra that leads to Vogan diagrams [Kna96b, Bat00, Bat02, CH04, CH06]. In our thesis we have taken the later approach.

3.3 Vogan diagrams

Let $\hat{\mathfrak{g}}_{\mathbb{R}}$ be an almost compact real form of $\hat{\mathfrak{g}}$ and σ be the Cartan involution on $\hat{\mathfrak{g}}_{\mathbb{R}}$ leading to the Cartan decomposition $\hat{\mathfrak{g}}_{\mathbb{R}} = \hat{\mathfrak{k}}_0 \oplus \hat{\mathfrak{p}}_0$. Let $\hat{\mathfrak{h}}_0 = t_0 \oplus a_0$ be the maximally compact σ-stable Cartan subalgebra of $\hat{\mathfrak{g}}_{\mathbb{R}}$ with complexification $\hat{\mathfrak{h}} = t \oplus a$. Let us denote $\hat{\Delta} = \hat{\Delta}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ be the set of roots of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$. This set doesn't contain any real root as $\hat{\mathfrak{h}}_0$ is assumed to be maximally compact. From $\hat{\Delta}$ we choose a positive system $\hat{\Delta}_+$ that takes it_0 before a_0. Since σ is $+1$ on t_0 and -1 on a_0 and since there are no real roots

$\sigma(\hat{\Delta}_+) = \hat{\Delta}_+$. Therefore σ permutes the simple roots. It fixes the simple roots that are imaginary and permutes in 2-cycles the simple roots that are complex.

Definition 3.3.1. *By Vogan diagram of the triple* $(\hat{\mathfrak{g}}_{\mathbb{R}}, \hat{\mathfrak{h}}_0, \hat{\Delta}_+)$ *we mean the Dynkin diagram of* $\hat{\Delta}^+$ *with the* 2*-element orbits under* σ *labelled an arrow and with the* 1*-element orbit painted or not depending upon whether the corresponding imaginary simple root is non-compact or compact.*

Every Vogan diagram represents an almost compact(non-compact) real form of some affine Kac-Moody Lie algebra. Two diagrams may represent isomorphic algebras and in that case the diagrams are equivalent. So the classification of Vogan diagram gives rise to the classification of almost compact real form of affine Kac-Moody Lie algebra.

The equivalence of Vogan diagram is defined as the equivalence relation generated by the following two operations:

1. applications of an automorphism of the Dynkin diagram;
2. change in the positive system by reflection in a simple, non-compact root, i.e., by a vertex which is colored in the Vogan diagram.

As a consequence of reflection by a simple non-compact root α, the rules for single and triple lines is that we have α colored and its immediate neighbour is changed to the opposite color. The rule for double line is that if α is the smaller root, then there is no change in the color of immediate neighbour, but we leave α colored. If α is a bigger root, then we leave α colored and the immediate neighbour is changed to the opposite color.

If two Vogan diagrams aren't equivalent to each other, then they are called non-equivalent.

Definition 3.3.2. *An abstract Vogan diagram is an irreducible abstract Dynkin diagram of non-twisted affine Kac-Moody Lie algebra with two additional piece of structure as follows:*

1. *one is an automorphism of order* 1 *or* 2 *of the diagram, which is indicated by labelling the* 2*-element orbits;*
2. *second one is a subset of* 1*-element orbits which is to be indicated by pointing the vertices corresponding to the members of the subset.*

Every Vogan diagram is an abstract Vogan diagram. Here we state some important results related to Vogan diagrams.

Theorem 3.3.3. *If an abstract Vogan diagram for an non-twisted affine Kac-Moody Lie algebra is given, then there exists an almost compact real form of a non-twisted affine Kac-Moody Lie algebra such that the given diagram is the Vogan diagram of this almost compact real form.*

Theorem 3.3.4. *If two almost compact real forms of a non-twisted affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ have equivalent Vogan diagram then they are isomorphic.*

It is always convenient to represent equivalence class of Vogan diagrams with minimum number of vertices painted. In this regard we have Borel Seibenthal theorem for affine Kac-Moody algebras [CH06].

Theorem 3.3.5. *Every equivalence class of Vogan diagram has a representative with atmost two vertices painted.*

Now we briefly review the definition and geometry of the affine Kac-Moody symmetric spaces [Pop05, Hei06, Fre09].

3.4 Affine symmetric spaces

We know that a compact irreducible symmetric space is either a compact simple Lie group G or a quotient G/K of a compact simple Lie group by the fixed point set of an involution ρ (or an open subgroup of it). If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of the Lie algebra $\mathfrak{g}$ of group G into $+1$ and -1 eigen spaces of ρ then K acts on $\mathfrak{g}$ by adjoint representation leaving the decomposition invariant. The restriction of this action to $\mathfrak{p}$ can be identified with the isotropy representation of G/K and we know that the isotropy representation of a symmetric space is polar i.e., there exists a linear subspace $\sum \subset \mathfrak{p}$ which meets every orbit and always orthogonally. In particular, $\sum$ is any maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ [Hel79, Loo69].

First we will describe Kac-Moody group [Fre09] to define the infinite version of the symmetric spaces.

3.4.1 Loop group

We know that the loop algebra associated to a Lie algebra $\mathfrak{g}$ is the following space:

$$\mathfrak{g}_{\text{loop}} := \{u : S^1 \to \mathfrak{g} \mid u(t+2\pi) = \sigma u(t)\} \tag{3.4.1}$$

for $\sigma \in \text{Aut}(\mathfrak{g}), \sigma^l = id$ and $u(t) = \sum_{n\in N} u_n e^{int/l}$ where $u_n \in \mathfrak{g}$. Similarly for compact, simply connected Lie group G with Lie algebra $\mathfrak{g}$ the loop group can be defined as

$$G_{\text{loop}} := \{g : \mathbb{R} \to G \mid g(t+2\pi) = \sigma g(t), g \in \mathbb{C}^\infty\} \tag{3.4.2}$$

where $\sigma : G \to G$.

3.4.2 Kac-Moody group

Kac-Moody groups can be constructed in two steps:

1. First we will construct an S^1- bundle in the real case (a $\mathbb{C}^*$-bundle in the complex case). This corresponds via the exponential map to the central term $\mathbb{R}c$ (resp. $\mathbb{C}c$) of the Kac-Moody algebra.

$$0 \to S^1 \to \bar{G} \to G_{\text{loop}} \to 0 \tag{3.4.3}$$

2. Then we construct a semi direct product with S^1- (resp. $\mathbb{C}^*$). This corresponds via the exponential map to $\mathbb{R}d$ (resp. $\mathbb{C}d$) of the Kac-Moody algebra.

We have to define $\bar{G}$ in such a way that its tangential Lie algebra at $e \in \bar{G}$ is isomorphic to $\bar{\mathfrak{g}}$.

3.4.3 Kac-Moody symmetric space

An affine Kac-Moody symmetric space is by definition either an affine Kac-Moody group $\hat{G}$ (group type) or a quotient $\hat{G}/\hat{K}$ where $\hat{K}$ is the fixed point set of an involution of the second kind $\hat{\rho}$. In fact, if $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{p}}$ is the splitting of the affine Lie algebra of $\hat{G}$ into the ± 1 eigenspaces of $\hat{\rho}$ then the metric of $\hat{G}/\hat{K}$ is the left invariant metric obtained from the restriction of the inner product of $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{p}}$. In finite dimension, the isotropy representation is polar itself while in infinite dimension it leaves invariant a co-dimension-2 submanifold which can be identified with a (pre-)Hilbert space and the induced action on this space is a polar action by affine isometries.

Geometrically an(affine) Kac-Moody symmetric space M is a tame Fr*é*chet Lorentz symmetric space [Fre09] such that its isometry group $I(M)$ contains a transitive subgroup isomorphic to an affine Kac-Moody group $\hat{G}$ and the intersection of the isotropy group of a point with $\hat{G}$ is a loop group of compact type.

We can distinguish Kac-Moody symmetric spaces as of Euclidean, compact and non-compact type, corresponding to the respective types of Riemannian symmetric spaces. The detail definition, existence theorem can be found in [Fre09].

3.5 Affine Kac-Moody symmetric spaces associated with $A_1^{(1)}, A_2^{(1)}$ and $A_2^{(2)}$

Let $\mathfrak{g}$ be a complex affine Kac-Moody algebra. Similarly like a finite dimensional complex Lie algebra (FSLA) a mapping $T : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$T([x,y]) = [T(x), T(y)], T(x+y) = T(x) + T(y), T(\alpha x) = \alpha^* T(x)$$

and $T^2 = Id$ for all $x, y \in \mathfrak{g}$ and for $\alpha \in \mathbb{C}$ is called a conjugation(involution) of $\mathfrak{g}$. For $\mathfrak{g}$ is a complex algebra and T be a conjugation of $\mathfrak{g}$, then $\mathfrak{g}_{\mathbb{R}} = \{x \in \mathfrak{g} : T(x) = x\}$ is a real form of $\mathfrak{g}$. A real algebra is called compact if its killing form is negative definite. Every complex affine Kac-Moody algebra $\mathfrak{g}$ has a compact real form(unique). Now starting with the compact real form $\mathfrak{u}$ we can construct all the various non-compact real forms by applying involutive automorphisms to $\mathfrak{u}$ followed by Weyl unitary trick.

3.5.1 Affine Kac-Moody symmetric spaces associated with $A_1^{(1)}$

The positive root system Δ_+ of $A_1^{(1)}$ is

$$\Delta_+(A_1^{(1)}) = \{\alpha, n\delta \pm \alpha, n\delta \mid n = 1, 2, \ldots\}$$

and the total root system is given by

$$\Delta(A_1^{(1)}) = \Delta_+(A_1^{(1)}) \cup \, (-\Delta_+(A_1^{(1)})).$$

Let $e_{\pm\gamma}$ for $\forall \gamma \in \Delta_+(A_1^{(1)})$ be constructed such that $e^*_{\pm\gamma} = e_{\mp\gamma}$ (and moreover $[e_\gamma, e_{-\gamma}] = h_\gamma, h^*_\gamma = h_\gamma$) where * is the Cartan anti-involution (antilinear antiautomorphism: $[x,y]^* = [y^*, x^*]$). Then the compact real form is generated as a real linear space by the elements:

$$\begin{aligned} &ih_\alpha,\; i(e_\alpha + e_{-\alpha}),\; (e_\alpha - e_{-\alpha}),\; i(e_{n\delta\pm\alpha} + e_{-n\delta\pm\alpha}), \\ &(e_{n\delta\pm\alpha} - e_{-n\delta\pm\alpha}), i(e^{(\alpha)}_{n\delta} + e^{(\alpha)}_{-n\delta}),\; (e^{(\alpha)}_{n\delta} - e^{(\alpha)}_{-n\delta}) \end{aligned} \tag{3.5.1}$$

These basis elements satisfy the condition $x^* + x = 0$.

The minimal evaluation representations of $A_1^{(1)}$: Let E_{ij} be the 2×2-matrix with 1 is in the (i, j)-th entry and the other entries are 0. We set

$$\begin{aligned} &h_\alpha = E_{11} - E_{22},\; e_\alpha = E_{12},\; e_{-\alpha} = E_{21},\; e^{(\alpha)}_{\pm n\delta} = t^{\pm n}(E_{11} - E_{22}), \\ &e_{\pm n\delta+\alpha} = t^{\pm n}E_{12},\; e_{\pm n\delta-\alpha} = t^{\pm n}E_{21}, \end{aligned}$$

for $n = 1, 2, \ldots$. This is the minimal matrix representations of $A_1^{(1)}$ without the central charge c and the derivation term d.

If we substitute this realization in (3.5.1) we obtain the basis elements of the compact real form for $A_1^{(1)}$. Explicitly compact real form is given by

$$\begin{aligned} \mathfrak{u}(t) = \sum_{n\in\mathbb{Z}_+} [&ia_1^{(n)}h_\alpha + ia_2^{(n)}(e_\alpha + e_{-\alpha}) + a_3^{(n)}(e_\alpha - e_{-\alpha}) + ia_4^{(n)}(e_{n\delta+\alpha} + e_{-n\delta-\alpha}) \\ &+ a_5^{(n)}(e_{n\delta+\alpha} - e_{-n\delta-\alpha}) + ia_6^{(n)}(e_{n\delta-\alpha} + e_{-n\delta+\alpha}) + a_7^{(n)}(e_{n\delta-\alpha} - e_{-n\delta+\alpha}) \\ &+ ia_8^{(n)}(e^{(\alpha)}_{n\delta} + e^{(\alpha)}_{-n\delta}) + a_9^{(n)}(e^{(\alpha)}_{n\delta} - e^{(\alpha)}_{-n\delta})] \oplus \mathbb{R}ic \oplus \mathbb{R}id \end{aligned}$$

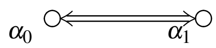
which is equal to the following matrix

$$\mathfrak{u}(t) = \sum_{n\in\mathbb{Z}_+}$$

$$\begin{pmatrix} ia_1 + ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(t^n - t^{-n}) & \begin{array}{c} ia_2 + a_3 + t^n(a_5^{(n)} + ia_4^{(n)}) + \\ t^{-n}(-a_7^{(n)} + ia_6^{(n)}) \end{array} \\ \begin{array}{c} ia_2 - a_3 + t^{-n}(-a_5^{(n)} + ia_4^{(n)}) + \\ t^n(a_7^{(n)} + ia_6^{(n)}) \end{array} & -ia_1 - ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(-t^n + t^{-n}) \end{pmatrix}$$

$$\oplus \mathbb{R}ic \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}id \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is a skew hermitian matrix with trace zero, which is identified as $\mathfrak{su}^{(1)}(2)$. The Vogan diagram associated with this real form is given by the following: Now the general



form [Cor92a] of an involutive automorphism associated with a affine Kac-Moody algebra is given as: for type 1(a) automorphism

$$\sigma(\mathfrak{u}(t)) = U(t)\mathfrak{u}(ut)U(t)^{-1} + \frac{1}{\gamma} Res\left\{ tr\left(U(t)^{-1}\frac{dU(t)}{dt}\mathfrak{u}(u(t))\right)\right\} c. \tag{3.5.2}$$

and for type 1(b) automorphism

$$\sigma(\mathfrak{u}(t)) = U(t)(-\tilde{\mathfrak{u}}(ut))U(t)^{-1} + \frac{1}{\gamma} Res\left\{ tr\left(U(t)^{-1}\frac{dU(t)}{dt}(-\tilde{\mathfrak{u}}(ut))\right)\right\} c. \tag{3.5.3}$$

But the conjugacy class of type 1(b) automorphisms with $u = 1$ and $u = -1$ correspond to some automorphisms of type 1(a) with $u = 1$ and $u = -1$ respectively. Also we shall like to mention that type 2(a) and type 2(b) automorphisms are obtained by composing type 1(a) and 1(b) with Cartan involution respectively. Action of σ on c is $\sigma(c) = \mu c$, however for 1(a) automorphism $\mu = 1$. Now $\sigma(d) = \mu\Phi(U(t)) + \xi c + \mu d$ where $\Phi(U(t))$ is the $d_\Gamma \times d_\Gamma$ matrix that depends upon $U(t)$ as below,

$$\Phi(U(t)) = \left\{ -t\frac{dU(t)}{dt}U(t)^{-1} + \frac{1}{d_\Gamma} tr\left(t\frac{dU(t)}{dt}U(t)^{-1}\right) I\right\}$$

and for our cases $\sigma(d) = d$ except the case-III of $A_1^{(1)}$ where

$$\sigma(d) = \begin{pmatrix} t^2/2 & 0 \\ 0 & -t^2/2 \end{pmatrix} + d.$$

Case-I: If

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; u = 1, \quad \xi = 0, \tag{3.5.4}$$

then under the automorphism (3.5.2) with (3.5.4) a general matrix with block matrices A,B,C,D transforms as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Hence, the fixed subalgebra K of $\mathfrak{u}(t)$ is given by

$$K = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} ia_1 + ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(t^n - t^{-n}) & 0 \\ 0 & -ia_1 - ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(-t^n + t^{-n}) \end{pmatrix}$$

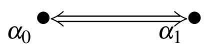
and

$$P = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} 0 & \begin{matrix} ia_2 + a_3 + t^n(a_5^{(n)} + ia_4^{(n)}) + \\ t^{-n}(-a_7^{(n)} + ia_6^{(n)}) \end{matrix} \\ \begin{matrix} ia_2 - a_3 + t^{-n}(-a_5^{(n)} + ia_4^{(n)}) + \\ t^n(a_7^{(n)} + ia_6^{(n)}) \end{matrix} & 0 \end{pmatrix}.$$

Thus

$$K + iP = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} ia_1 + ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(t^n - t^{-n}) & \begin{matrix} -a_2 + ia_3 + it^n(a_5^{(n)} + ia_4^{(n)}) + \\ it^{-n}(-a_7^{(n)} + ia_6^{(n)}) \end{matrix} \\ \begin{matrix} -a_2 - ia_3 + it^{-n}(-a_5^{(n)} + ia_4^{(n)}) + \\ it^n(a_7^{(n)} + ia_6^{(n)}) \end{matrix} & -ia_1 - ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(-t^n + t^{-n}) \end{pmatrix}.$$

Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in su_1^{(1)}(1,1)$ and the corresponding Vogan diagram is given by



The two affine Kac-Moody symmetric spaces are

$$SU_1^{(1)}(1,1)/S_1^{(1)}(U_1 \times U_1), \quad SU^{(1)}(1+1)/S_1^{(1)}(U_1 \times U_1). \tag{3.5.5}$$

Case II: Similarly if

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = -1, \quad \xi = 0 \tag{3.5.6}$$

then under the automorphism(3.5.2) with (3.5.6) the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ trasforms as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} -A & B \\ C & -D \end{pmatrix}.$$

We observe when n is an even integer it reduces to Case-I, giving same real form. But if

n is odd then we have

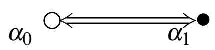
$$K = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} 0 & \begin{matrix} ia_2 + a_3 + t^n(a_5^{(n)} + ia_4^{(n)}) + \\ t^{-n}(-a_7^{(n)} + ia_6^{(n)}) \end{matrix} \\ \begin{matrix} ia_2 - a_3 + t^{-n}(-a_5^{(n)} + ia_4^{(n)}) + \\ t^n(a_7^{(n)} + ia_6^{(n)}) \end{matrix} & 0 \end{pmatrix}$$

and

$$P = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} ia_1 + ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(t^n - t^{-n}) & 0 \\ 0 & -ia_1 - ia_8^{(n)}(t^n + t^{-n}) + a_9^{(n)}(-t^n + t^{-n}) \end{pmatrix}.$$

$$K + iP = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} -a_1 - a_8^{(n)}(t^n + t^{-n}) + ia_9^{(n)}(t^n - t^{-n}) & \begin{matrix} ia_2 + a_3 + t^n(a_5^{(n)} + ia_4^{(n)}) + \\ t^{-n}(-a_7^{(n)} + ia_6^{(n)}) \end{matrix} \\ \begin{matrix} ia_2 - a_3 + t^{-n}(-a_5^{(n)} + ia_4^{(n)}) + \\ t^n(a_7^{(n)} + ia_6^{(n)}) \end{matrix} & a_1 + a_8^{(n)}(t^n + t^{-n}) + ia_9^{(n)}(-t^n + t^{-n}) \end{pmatrix}.$$

Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in \mathfrak{su}_{-1}^{(1)}(1,1)$. The Vogan diagram is the given by .



Therefore, the corresponding affine Kac-Moody symmetric spaces are:

$$SU_{-1}^{(1)}(1,1)/S_{-1}^{(1)}(U_1 \times U_1), \quad SU^{(1)}(1+1)/S_{-1}^{(1)}(U_1 \times U_1). \tag{3.5.7}$$

Case III: Now consider

$$U(t) = \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix}, \quad u = 1, \quad \xi = -1, \quad U(t)^{-1} = \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix}. \tag{3.5.8}$$

So now under the automorphism (3.5.2) with (3.5.8) a general matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ transform as,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} D & -Ct^{-1} \\ -Bt & A \end{pmatrix}. \tag{3.5.9}$$

Then from a simple mathematical manipulation we observe that in this case

$$K = \frac{1}{2}\begin{pmatrix} A + D & B - Ct^{-1} \\ C - Bt & D + A \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} A - D & B + Ct^{-1} \\ C + Bt & D - A. \end{pmatrix} \tag{3.5.10}$$

Now putting the values of A, B, C and D in K we get $K = \frac{1}{2} \sum_{n \in \mathbb{Z}_+}$

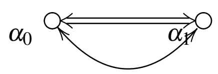
$$\begin{pmatrix} 0 & \begin{array}{c} ia_2^{(n)} + a_3^{(n)} + t^n(ia_4^{(n)} + a_5^{(n)}) + \\ t^{-n}(ia_6^{(n)} - a_7^{(n)}) + -ia_2^{(n)} t^{-1} + a_3^{(n)} t^{-1} + \\ t^{-(n+1)}(-ia_4^{(n)} + a_5^{(n)}) - (ia_6^{(n)} + a_7^{(n)}) t^{n-1} \end{array} \\ \begin{array}{c} ia_2^{(n)} - a_3^{(n)} + t^{-n}(ia_4^{(n)} - a_5^{(n)}) + \\ t^n(ia_6^{(n)} + a_7^{(n)}) - ia_2^{(n)} t - a_3^{(n)} t + \\ t^{n+1}(-ia_4^{(n)} - a_5^{(n)}) + (-ia_6^{(n)} + a_7^{(n)}) t^{-n+1} \end{array} & 0 \end{pmatrix}$$

with K satisfies $K^* + K = 0$ and hence K is identified as $\mathfrak{so}^{(1)}(2)$. So we have $K + iP = \frac{1}{2} \sum_{n \in \mathbb{Z}_+}$

$$\begin{pmatrix} -2a_1^{(n)} - 2a_8^{(n)}(t^n + t^{-n}) + 2ia_9^{(n)}(t^n - t^{-n}) & \begin{array}{c} ia_2^{(n)} + a_3^{(n)} + t^n(ia_4^{(n)} + a_5^{(n)}) + \\ t^{-n}(ia_6^{(n)} - a_7^{(n)}) + -ia_2^{(n)} t^{-1} + a_3^{(n)} t^{-1} + \\ t^{-(n+1)}(-ia_4^{(n)} + a_5^{(n)}) - (ia_6^{(n)} + a_7^{(n)}) t^{n-1} \end{array} \\ \begin{array}{c} ia_2^{(n)} - a_3^{(n)} + t^{-n}(ia_4^{(n)} - a_5^{(n)}) + \\ t^n(ia_6^{(n)} + a_7^{(n)}) - ia_2^{(n)} t - a_3^{(n)} t + \\ t^{n+1}(-ia_4^{(n)} - a_5^{(n)}) + (-ia_6^{(n)} + a_7^{(n)}) t^{-n+1} \end{array} & 2a_1^{(n)} + 2a_8^{(n)}(t^n + t^{-n}) + 2ia_9^{(n)}(t^{-n} - t^n) \end{pmatrix}$$

$$\oplus \mathbb{R}ic \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{R}id \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is identified as $\mathfrak{sl}^{(1)}(2, \mathbb{R})$ and the Vogan diagram is



Hence the affine Kac-Moody symmetric spaces are

$$SU^{(1)}(2)/SO^{(1)}(2), \quad SL^{(1)}(2, \mathbb{R})/SO^{(1)}(2). \tag{3.5.11}$$

3.5.2 Affine Kac-Moody symmetric spaces associated with $A_2^{(1)}$

The positive root system Δ_+ of $A_2^{(1)}$ is

$$\Delta_+(A_2^{(1)}) = \{\alpha, \beta, \alpha+\beta, n\delta \pm \alpha, n\delta \pm \beta, n\delta \pm (\alpha+\beta), n\delta \mid n = 1, 2, \ldots\}$$

and the total root system is given by

$$\Delta(A_2^{(1)}) = \Delta_+(A_2^{(1)}) \cup (-\Delta_+(A_2^{(1)})).$$

Let $e_{\pm\gamma}$ for $\forall \gamma \in \Delta_+(A_2^{(1)})$ be constructed such that $e_{\pm\gamma}^* = e_{\mp\gamma}$. Then the compact real form is generated as a real linear space by the elements:

$$ih_\alpha,\ ih_\beta,\ i(e_\gamma + e_{-\gamma}),\ (e_\gamma - e_{-\gamma}),\ i(e_{n\delta\pm\gamma} + e_{-n\delta\pm\gamma}),\ (e_{n\delta\pm\gamma} - e_{-n\delta\pm\gamma}), \tag{3.5.12}$$

$$i(e^{(\alpha)}_{n\delta}+e^{(\alpha)}_{-n\delta}),\ (e^{(\alpha)}_{n\delta}-e^{(\alpha)}_{-n\delta}),\ i(e^{(\beta)}_{n\delta}+e^{(\beta)}_{-n\delta}),\ (e^{(\beta)}_{n\delta}-e^{(\beta)}_{-n\delta}) \tag{3.5.13}$$

for $\gamma=\alpha,\beta,\alpha+\beta; n=1,2,\ldots$. These basis elements satisfy the condition $x^*+x=0$.

The minimal evaluation representations of $A^{(1)}_2$: Let E_{ij} be the 3×3-matrix with 1 is in the (i,j)-th entry and the other entries are 0. We set

$$\begin{aligned}
&h_\alpha=E_{11}-E_{22},\ e_\alpha=E_{12},\ e_{-\alpha}=E_{21},\\
&h_\beta=E_{22}-E_{33},\ e_\beta=E_{23},\ e_{-\beta}=E_{32},\\
&e_{\alpha+\beta}=E_{13},\ e_{-\alpha-\beta}=E_{31},\\
&e^{(\alpha)}_{\pm n\delta}=t^{\pm n}(E_{11}-E_{22}),\ e^{(\beta)}_{\pm n\delta}=t^{\pm n}(E_{22}-E_{33}),\\
&e_{\pm n\delta+\alpha}=t^{\pm n}E_{12},\ e_{\pm n\delta-\alpha}=t^{\pm n}E_{21},\\
&e_{\pm n\delta+\beta}=t^{\pm n}E_{23},\ e_{\pm n\delta-\beta}=t^{\pm n}E_{32},\\
&e_{\pm n\delta+\alpha+\beta}=t^{\pm n}E_{13},\ e_{\pm n\delta-\alpha-\beta}=t^{\pm n}E_{31},
\end{aligned}$$

for $n=1,2,\ldots$. This is the minimal matrix representations of $A^{(1)}_2$ without the central charge c and the derivation term d.

If we substitute this realization in (3.5.12) we obtain the basis elements of the compact real form for $A^{(1)}_2$. Explicitly compact real form is given by

$$\begin{aligned}
\mathfrak{u}(t) \;=\; &\sum_{n\in\mathbb{Z}_+}[ia^{(n)}_1h_\alpha+ia^{(n)}_2h_\beta+ia^{(n)}_3(e_\alpha+e_{-\alpha})+a^{(n)}_4(e_\alpha-e_{-\alpha})+ia^{(n)}_5(e_\beta+e_{-\beta})\\
+\;& a^{(n)}_6(e_\beta-e_{-\beta})+ia^{(n)}_7(e_{\alpha+\beta}+e_{-\alpha-\beta})+a^{(n)}_8(e_{\alpha+\beta}-e_{-\alpha-\beta})\\
+\;& ia^{(n)}_9(e_{n\delta+\alpha}+e_{-n\delta-\alpha})+a^{(n)}_{10}(e_{n\delta+\alpha}-e_{-n\delta-\alpha})+ia^{(n)}_{11}(e_{n\delta-\alpha}+e_{-n\delta+\alpha})\\
+\;& a^{(n)}_{12}(e_{n\delta-\alpha}-e_{-n\delta+\alpha})+ia^{(n)}_{13}(e_{n\delta+\beta}+e_{-n\delta-\beta})+a^{(n)}_{14}(e_{n\delta+\beta}-e_{-n\delta-\beta})\\
+\;& ia^{(n)}_{15}(e_{n\delta-\beta}+e_{-n\delta+\beta})+a^{(n)}_{16}(e_{n\delta-\beta}-e_{-n\delta+\beta})+ia^{(n)}_{17}(e_{n\delta+\alpha+\beta}+e_{-n\delta-\alpha-\beta})\\
+\;& a^{(n)}_{18}(e_{n\delta+\alpha+\beta}-e_{-n\delta-\alpha-\beta})+ia^{(n)}_{19}(e_{n\delta-\alpha-\beta}+e_{-n\delta+\alpha+\beta})+a^{(n)}_{20}(e_{n\delta-\alpha-\beta}\\
-\;& e_{-n\delta+\alpha+\beta})+ia^{(n)}_{21}(e^{(\alpha)}_{n\delta}+e^{(\alpha)}_{-n\delta})+a^{(n)}_{22}(e^{(\alpha)}_{n\delta}-e^{(\alpha)}_{-n\delta})+ia^{(n)}_{23}(e^{(\beta)}_{n\delta}+e^{(\beta)}_{-n\delta})\\
+\;& a^{(n)}_{24}(e^{(\beta)}_{n\delta}-e^{(\beta)}_{-n\delta})]\oplus\mathbb{R}ic\oplus\mathbb{R}id
\end{aligned}$$

So, the compact form(in matrix realization) $\mathfrak{u}(t)$ is

$$\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}A^{(n)}_{11} & A^{(n)}_{12} & A^{(n)}_{13}\\ A^{(n)}_{21} & A^{(n)}_{22} & A^{(n)}_{23}\\ A^{(n)}_{31} & A^{(n)}_{32} & A^{(n)}_{33}\end{pmatrix}\oplus\mathbb{R}ic\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\oplus\mathbb{R}id\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

where

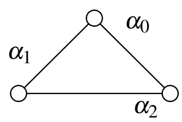
$$A^{(n)}_{11} \;=\; ia^{(n)}_1+ia^{(n)}_{21}(t^n+t^{-n})+a^{(n)}_{22}(t^n-t^{-n})$$

$$
\begin{aligned}
A_{12}^{(n)} &= ia_3^{(n)} + a_4^{(n)} + (ia_9^{(n)} + a_{10}^{(n)})t^n + (ia_{11}^{(n)} - a_{12}^{(n)})t^{-n} \\
A_{13}^{(n)} &= ia_7^{(n)} + a_8^{(n)} + (ia_{17}^{(n)} + a_{18}^{(n)})t^n + (ia_{19}^{(n)} - a_{20}^{(n)})t^{-n} \\
A_{21}^{(n)} &= ia_3^{(n)} - a_4^{(n)} + (ia_{11}^{(n)} + a_{12}^{(n)})t^n + (ia_9^{(n)} - a_{10}^{(n)})t^{-n} \\
A_{22}^{(n)} &= -ia_1^{(n)} + ia_2^{(n)} - ia_{21}^{(n)}(t^n + t^{-n}) + a_{22}^{(n)}(t^{-n} - t^n) + ia_{23}^{(n)}(t^n + t^{-n}) + a_{24}^{(n)}(t^n - t^{-n}) \\
A_{23}^{(n)} &= ia_5^{(1)} + a_6^{(n)} + (ia_{13}^{(n)} + a_{14}^{(n)})t^n + (ia_{15}^{(n)} - a_{16}^{(n)})t^{-n} \\
A_{31}^{(n)} &= ia_7^{(n)} - a_8^{(n)} + (ia_{17}^{(n)} a_{18}^{(n)})t^{-n} + (ia_{19}^{(n)} + a_{20}^{(n)})t^n \\
A_{32}^{(n)} &= ia_5^{(n)} - a_6^{(n)} + (ia_{13}^{(n)} - a_{14}^{(n)})t^{-n} + (ia_{15}^{(n)} + a_{16}^{(n)})t^n \\
A_{33}^{(n)} &= -ia_2^{(n)} - ia_{23}^{(n)}(t^n + t^{-n}) + a_{24}^{(n)}(t^{-n} - t^n)
\end{aligned}
$$

This matrix is in the form:

$$
\begin{pmatrix} (A)_{2\times 2} & (B)_{2\times 1} \\ (-B^*)_{1\times 2} & (C)_{1\times 1} \end{pmatrix}, \tag{3.5.14}
$$

which is a skew hermitian matrix with trace zero it is identified as $\mathfrak{su}^{(1)}(3)$. The Vogan diagram is



Now proceeding similarly as $A_1^{(1)}$ case and taking different cases we have,

Case I:

$$
U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad u = 1, \quad \xi = 0. \tag{3.5.15}
$$

Under the automorphism (3.5.2) with (3.5.15) the matrix transforms as

$$
\begin{pmatrix} A & B \\ (-B^*) & C \end{pmatrix} \longrightarrow \begin{pmatrix} A & -B \\ B^* & C \end{pmatrix} \tag{3.5.16}
$$

Here

$$
K = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad P = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}. \tag{3.5.17}
$$

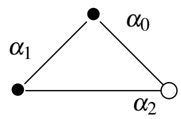
$$
\begin{aligned}
K &= \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \\
&= \begin{pmatrix} A - \frac{1}{2}Tr(A)I_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}Tr(A)I_2 & 0 \\ 0 & Tr(C) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - Tr(C) \end{pmatrix}
\end{aligned} \tag{3.5.18}
$$

shows that K is isomorphic to $\mathfrak{su}(2) \times c_0 \times \mathfrak{su}(1)$ where c_0 is the center of K. Now $K + iP$

is

$$\begin{pmatrix} (Z_1)_{2\times 2} & (Z_2)_{2\times 1} \\ (Z_2^*)_{1\times 2} & (Z_3)_{1\times 1} \end{pmatrix} \tag{3.5.19}$$

with $Z_2 = iB$ is a 2×2 matrix, $Z_1 = A$ is a 2×2 skew hermitian matrix and $Z_3 = C$ is a 1×1 skew hermitian matrix and also satisfies $TrZ_1 + TrZ_3 = 0$. Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in \mathfrak{su}_1^{(1)}(2,1)$. The Vogan diagram is



Thus the corresponding symmetric spaces are:

$$SU_1^{(1)}(2,1)/S_1^{(1)}(U_2\times U_1), \quad SU^{(1)}(2+1)/S_1^{(1)}(U_2\times U_1). \tag{3.5.20}$$

Case II:

$$U(t) = U(t)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad u = -1, \quad \xi = 0. \tag{3.5.21}$$

Under the automorphism (3.5.2) with (3.5.21) the matrix transforms as

$$\begin{pmatrix} A & B \\ -B^* & C \end{pmatrix} \longrightarrow \begin{pmatrix} -A & B \\ -B^* & -C \end{pmatrix}. \tag{3.5.22}$$

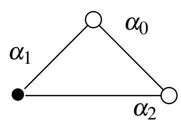
Here

$$K = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \quad P = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \tag{3.5.23}$$

Thus $K + iP$ is

$$\begin{pmatrix} (Z_1)_{2\times 2} & (Z_2)_{2\times 1} \\ (-Z_2^*)_{1\times 2} & (Z_3)_{1\times 1} \end{pmatrix} \tag{3.5.24}$$

with $Z_2 = iB$ is a 2×1 matrix, $Z_1 = A$ is a 2×2 hermitian matrix and $Z_3 = C$ is a 1×1 hermitian matrix and also satisfies $TrZ_1 + TrZ_3 = 0$. Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in su_{-1}^{(1)}(2,1)$. The Vogan diagram is



Thus the corresponding symmetric spaces are:

$$SU_{-1}^{(1)}(2,1)/S_{-1}^{(1)}(U_2\times U_1), \quad SU^{(1)}(2+1)/S_{-1}^{(1)}(U_2\times U_1). \tag{3.5.25}$$

Case III:

$$U(t)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad u=1,\quad \xi=0,\quad U(t)^{-1}=\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}. \tag{3.5.26}$$

Under the automorphism (3.5.2) with (3.5.26) the matrix transforms as

$$\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}A_{11}^{(n)}&A_{12}^{(n)}&A_{13}^{(n)}\\A_{21}^{(n)}&A_{22}^{(n)}&A_{23}^{(n)}\\A_{31}^{(n)}&A_{32}^{(n)}&A_{33}^{(n)}\end{pmatrix}\longrightarrow\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}A_{33}^{(n)}&A_{32}^{(n)}&A_{31}^{(n)}\\A_{23}^{(n)}&A_{22}^{(n)}&A_{21}^{(n)}\\A_{13}^{(n)}&A_{12}^{(n)}&A_{11}^{(n)}\end{pmatrix}. \tag{3.5.27}$$

$$\begin{aligned}K&=\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}A_{11}^{(n)}+A_{33}^{(n)}&A_{12}^{(n)}+A_{32}^{(n)}&A_{13}^{(n)}+A_{31}^{(n)}\\A_{21}^{(n)}+A_{23}^{(n)}&2A_{22}^{(n)}&A_{23}^{(n)}+A_{21}^{(n)}\\A_{31}^{(n)}+A_{13}^{(n)}&A_{32}^{(n)}+A_{12}^{(n)}&A_{33}^{(n)}+A_{11}^{(n)}\end{pmatrix}\\&=\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}(\widehat{A}_{11}^{(n)})_+&(\widehat{A}_{12}^{(n)})_+&(\widehat{A}_{13}^{(n)})_+\\(\widehat{A}_{21}^{(n)})_+&(\widehat{A}_{22}^{(n)})_+&(\widehat{A}_{23}^{(n)})_+\\(\widehat{A}_{31}^{(n)})_+&(\widehat{A}_{32}^{(n)})_+&(\widehat{A}_{33}^{(n)})_+\end{pmatrix}\end{aligned}$$

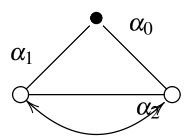
and $K\in\mathfrak{so}_1^{(1)}(3)$.

$$\begin{aligned}P&=\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}A_{11}^{(n)}-A_{33}^{(n)}&A_{12}^{(n)}-A_{32}^{(n)}&A_{13}^{(n)}-A_{31}^{(n)}\\A_{21}^{(n)}-A_{23}^{(n)}&0&A_{23}^{(n)}-A_{21}^{(n)}\\A_{31}^{(n)}-A_{13}^{(n)}&A_{32}^{(n)}-A_{12}^{(n)}&A_{33}^{(n)}-A_{11}^{(n)}\end{pmatrix}\\&=\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}(\widehat{A}_{11}^{(n)})_-&(\widehat{A}_{12}^{(n)})_-&(\widehat{A}_{13}^{(n)})_-\\(\widehat{A}_{21}^{(n)})_-&0&(\widehat{A}_{23}^{(n)})_-\\(\widehat{A}_{31}^{(n)})_-&(\widehat{A}_{32}^{(n)})_-&(\widehat{A}_{33}^{(n)})_-\end{pmatrix}.\end{aligned}$$

Thus $K+iP$ is

$$\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix}(\widehat{A}_{11}^{(n)})_++i(\widehat{A}_{11}^{(n)})_-&(\widehat{A}_{12}^{(n)})_++i(\widehat{A}_{12}^{(n)})_-&(\widehat{A}_{13}^{(n)})_++i(\widehat{A}_{13}^{(n)})_-\\(\widehat{A}_{21}^{(n)})_++i(\widehat{A}_{21}^{(n)})_-&(\widehat{A}_{22}^{(n)})_+&(\widehat{A}_{23}^{(n)})_++i(\widehat{A}_{23}^{(n)})_-\\(\widehat{A}_{31}^{(n)})_++i(\widehat{A}_{31}^{(n)})_-&(\widehat{A}_{32}^{(n)})_++i(\widehat{A}_{32}^{(n)})_-&(\widehat{A}_{33}^{(n)})_++i(\widehat{A}_{33}^{(n)})_-\end{pmatrix} \tag{3.5.28}$$

such that trace of this matrix is zero. Hence the non-compact real form is $K+iP\oplus\mathbb{R}ic\oplus\mathbb{R}id\in\mathfrak{sl}_1^{(1)}(3,\mathbb{R})$. The Vogan diagram is



Thus the corresponding symmetric spaces are

$$SL_1^{(1)}(3,\mathbb{R})/SO_1^{(1)}(3),\quad SU^{(1)}(3)/SO_1^{(1)}(3). \tag{3.5.29}$$

Case IV:

$$U(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad u = -1, \quad \xi = 0, \quad U(t)^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.5.30}$$

Under the automorphism (3.5.2) with (3.5.30) the matrix transforms as

$$\sum_{n\in\mathbb{Z}_+} \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \longrightarrow \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} -A_{33}^{(n)} & -A_{32}^{(n)} & -A_{31}^{(n)} \\ -A_{23}^{(n)} & -A_{22}^{(n)} & -A_{21}^{(n)} \\ -A_{13}^{(n)} & -A_{12}^{(n)} & -A_{11}^{(n)} \end{pmatrix}. \tag{3.5.31}$$

Here

$$\begin{aligned} K &= \frac{1}{2} \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} A_{11}^{(n)} - A_{33}^{(n)} & A_{12}^{(n)} - A_{32}^{(n)} & A_{13}^{(n)} - A_{31}^{(n)} \\ A_{21}^{(n)} - A_{23}^{(n)} & 0 & A_{23}^{(n)} - A_{21}^{(n)} \\ A_{31}^{(n)} - A_{13}^{(n)} & A_{32}^{(n)} - A_{12}^{(n)} & A_{33}^{(n)} - A_{11}^{(n)} \end{pmatrix} \\ &= \frac{1}{2} \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_- & (\widehat{A}_{13}^{(n)})_- \\ (\widehat{A}_{21}^{(n)})_- & 0 & (\widehat{A}_{23}^{(n)})_- \\ (\widehat{A}_{31}^{(n)})_- & (\widehat{A}_{32}^{(n)})_- & (\widehat{A}_{33}^{(n)})_- \end{pmatrix} \end{aligned}$$

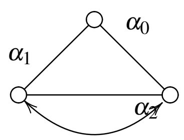
and we see that $K \in \mathfrak{so}_{-1}^{(1)}(3)$.

$$\begin{aligned} P &= \frac{1}{2} \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} A_{11}^{(n)} + A_{33}^{(n)} & A_{12}^{(n)} + A_{32}^{(n)} & A_{13}^{(n)} + A_{31}^{(n)} \\ A_{21}^{(n)} + A_{23}^{(n)} & 2A_{22}^{(n)} & A_{23}^{(n)} + A_{21}^{(n)} \\ A_{31}^{(n)} + A_{13}^{(n)} & A_{32}^{(n)} + A_{12}^{(n)} & A_{11}^{(n)} + A_{33}^{(n)} \end{pmatrix} \\ &= \frac{1}{2} \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_+ & (\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_+ \\ (\widehat{A}_{21}^{(n)})_+ & (\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_+ & (\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_+ \end{pmatrix} \end{aligned}$$

Thus $K + iP$ is

$$\frac{1}{2} \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} (\widehat{A}_{11}^{(n)})_- + i(\widehat{A}_{11}^{(n)})_+ & (\widehat{A}_{12}^{(n)})_- + i(\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_- + i(\widehat{A}_{13}^{(n)})_+ \\ (\widehat{A}_{21}^{(n)})_- + i(\widehat{A}_{21}^{(n)})_+ & i(\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_- + i(\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_- + i(\widehat{A}_{31}^{(n)})_+ & (\widehat{A}_{32}^{(n)})_- + i(\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_- + i(\widehat{A}_{33}^{(n)})_+ \end{pmatrix} \tag{3.5.32}$$

such that trace of this matrix is zero. Hence the non-compact real form is $K + iP \oplus \mathbb{R}ic \oplus \mathbb{R}id \in sl_{-1}^{(1)}(3, \mathbb{R})$. The Vogan diagram is



Thus the corresponding symmetric spaces are

$$SL_{-1}^{(1)}(3, \mathbb{R})/SO_{-1}^{(1)}(3), \quad SU^{(1)}(3)/SO_{-1}^{(1)}(3). \tag{3.5.33}$$

3.5.3 Affine Kac-Moody symmetric spaces associated with $A_2^{(2)}$

The positive root system Δ_+ of $A_2^{(2)}$ is

$$\Delta_+(A_2^{(2)}) = \{\alpha, n\delta \pm \alpha, (2n-1)\delta \pm 2\alpha, n\delta \mid n = 1,2,\ldots\}$$

and the total root system is given by

$$\Delta(A_2^{(2)}) = \Delta_+(A_2^{(2)}) \cup \ (-\Delta_+(A_2^{(2)})).$$

Let $e_{\pm\gamma}$ for $\forall\gamma \in \Delta_+(A_2^{(2)})$ be constructed such that $e^*_{\pm\gamma} = e_{\mp\gamma}$. Then the compact real form is generated as a real linear space by the elements:

$$ih_\alpha,\ i(e_\gamma + e_{-\gamma}),\ (e_\gamma - e_{-\gamma}) \text{ for } \forall\gamma \in \Delta_+(A_2^{(2)}). \tag{3.5.34}$$

These basis elements satisfy the condition $x^* + x = 0$.

The minimal evaluation representations of $A_2^{(2)}$: Let E_{ij} be the 3×3-matrix with 1 is in the (i,j)-th entry and the other entries are 0. We set

$$\begin{aligned}
&h_\alpha = E_{11} - E_{33},\ e_\alpha = E_{12} + E_{23},\ e_{-\alpha} = E_{21} + E_{32},\ e_{\pm 2n\delta} = t^{\pm 2n}(E_{11} - E_{33})\\
&e_{\pm 2n\delta+\alpha} = t^{\pm 2n}(E_{12} + E_{23}),\ e_{\pm 2n\delta-\alpha} = t^{\pm 2n}(E_{21} + E_{32}),\ e_{\pm(2n-1)\delta} = t^{\pm(2n-1)}\\
&(E_{11} - 2E_{22} + E_{33}),\ e_{\pm(2n-1)\delta+2\alpha} = t^{\pm(2n-1)}E_{13},\ e_{\pm(2n-1)\delta-2\alpha} = t^{\pm(2n-1)}E_{31}\\
&e_{\pm(2n-1)\delta+\alpha} = t^{\pm(2n-1)}(E_{12} - E_{23}),\ e_{\pm(2n-1)\delta-\alpha} = t^{\pm(2n-1)}(E_{21} - E_{32})
\end{aligned}$$

for $n = 1,2,\ldots$. This is the minimal matrix representations of $A_2^{(2)}$ without the central charge c and the derivation term d.

If we substitute this realization in (3.5.34) we obtain the basis elements of the compact real form for $A_2^{(2)}$. Explicitly compact real form is given by

$$\begin{aligned}
\mathfrak{u}(t) \ = \ & \sum_{n\in\mathbb{Z}_+} [ia_1^{(n)}h_\alpha + ia_2^{(n)}(e_\alpha + e_{-\alpha}) + a_3^{(n)}(e_\alpha - e_{-\alpha}) + ia_4^{(n)}(e_{2n\delta+\alpha} + e_{-2n\delta-\alpha})\\
+ \ & a_5^{(n)}(e_{2n\delta+\alpha} - e_{-2n\delta-\alpha}) + ia_6^{(n)}(e_{2n\delta-\alpha} + e_{-2n\delta+\alpha}) + a_7^{(n)}(e_{2n\delta-\alpha} - e_{-2n\delta+\alpha})\\
+ \ & ia_8^{(n)}(e_{(2n-1)\delta+\alpha} + e_{-(2n-1)\delta-\alpha}) + a_9^{(n)}(e_{(2n-1)\delta+\alpha} + e_{-(2n-1)\delta-\alpha})\\
+ \ & ia_{10}^{(n)}(e_{(2n-1)\delta-\alpha}e_{-(2n-1)\delta+\alpha}) + a_{11}^{(n)}(e_{(2n-1)\delta-\alpha} + e_{-(2n-1)\delta+\alpha})\\
+ \ & ia_{12}^{(n)}(e_{(2n-1)\delta+2\alpha} + e_{-(2n-1)\delta-2\alpha}) + a_{13}^{(n)}(e_{(2n-1)\delta+\alpha} - e_{-(2n-1)\delta-\alpha})\\
+ \ & ia_{14}^{(n)}(e_{(2n-1)\delta-2\alpha} + e_{-(2n-1)\delta+2\alpha}) + a_{15}^{(n)}(e_{(2n-1)\delta-\alpha} - e_{-(2n-1)\delta+\alpha})\\
+ \ & ia_{16}^{(n)}(e_{2n\delta} + e_{-2n\delta})a_{17}^{(n)}(e_{2n\delta} - e_{-2n\delta}) + ia_{18}^{(n)}(e_{(2n-1)\delta} + e_{-(2n-1)\delta})\\
+ \ & a_{19}^{(n)}(e_{(2n-1)\delta} - e_{-(2n-1)\delta})] \oplus \mathbb{R}ic \oplus \mathbb{R}id
\end{aligned}$$

So, the compact form is given by

$$\mathfrak{u}(t) = \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \oplus \mathbb{R}ic \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathbb{R}id \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned}
A_{11}^{(n)} &= ia_1^{(n)} + ia_{16}^{(n)}(t^{2n}+t^{-2n}) + a_{17}^{(n)}(t^{2n}-t^{-2n}) + ia_{18}^{(n)}(t^{2n-1}+t^{-2n+1}) \\
&+ a_{19}^{(n)}(t^{2n-1}-t^{-2n+1}) \\
A_{12}^{(n)} &= ia_2^{(n)} + a_3^{(n)} + (ia_4^{(n)}+a_5^{(n)})t^{2n} + (ia_6^{(n)}-a_7^{(n)})t^{-2n} + (ia_8^{(n)}+a_9^{(n)})t^{2n-1} \\
&+ (ia_{10}^{(n)}-a_{11}^{(n)})t^{-2n+1} \\
A_{13}^{(n)} &= (ia_{12}^{(n)}+a_{13}^{(n)})t^{2n-1} + (ia_{14}^{(n)}-a_{15}^{(n)})t^{-2n+1} \\
A_{21}^{(n)} &= ia_2^{(n)} - a_3^{(n)} + (ia_4^{(n)}-a_5^{(n)})t^{-2n} + (ia_6^{(n)}+a_7^{(n)})t^{2n} + (ia_8^{(n)}-a_9^{(n)})t^{-2n+1} \\
&+ (ia_{10}^{(n)}+a_{11}^{(n)})t^{2n-1} \\
A_{22}^{(n)} &= (ia_{18}^{(n)}+a_{19}^{(n)})(-2t^{2n-1}) + (a_{19}^{(n)}-ia_{18}^{(n)})(2t^{-2n+1}) \\
A_{23}^{(n)} &= ia_2^{(n)} + a_3^{(n)} + (ia_4^{(n)}+a_5^{(n)})t^{2n} + (ia_6^{(n)}-a_7^{(n)})t^{-2n} - (ia_8^{(n)}+a_9^{(n)})t^{2n-1} \\
&+ (a_{11}^{(n)}-ia_{10}^{(n)})t^{-2n+1} \\
A_{31}^{(n)} &= (ia_{12}^{(n)}-a_{13}^{(n)})t^{-2n+1} + (ia_{14}^{(n)}+a_{15}^{(n)})t^{2n-1} \\
A_{32}^{(n)} &= ia_2^{(n)} - a_3^{(n)} + (ia_4^{(n)}-a_5^{(n)})t^{-2n} + (ia_6^{(n)}+a_7^{(n)})t^{2n} + (a_9^{(n)}-ia_8^{(n)})t^{-2n+1} \\
&- (ia_{10}^{(n)}+a_{11}^{(n)})t^{2n-1} \\
A_{33}^{(n)} &= -ia_1^{(n)} - (ia_{16}^{(n)}+a_{17}^{(n)})t^{2n} + (a_{17}^{(n)}-ia_{16}^{(n)})t^{-2n} + (ia_{18}^{(n)}+a_{19}^{(n)})t^{2n-1} \\
&+ (ia_{18}^{(n)}-a_{19}^{(n)})t^{-2n+1}
\end{aligned}$$

which can be written in the form

$$\begin{pmatrix} (A)_{2\times 2} & (B)_{2\times 1} \\ (-B^*)_{1\times 2} & (C)_{1\times 1} \end{pmatrix} \tag{3.5.35}$$

which is a skew hermitian matrix with trace zero, hence it belongs to $su^{(2)}(3)$.

Case I:

$$U(t) = U(t)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad u = 1, \quad \xi = 0. \tag{3.5.36}$$

Under the automorphism (3.5.2) with (3.5.36) the matrix

$$\sum_{n\in\mathbb{Z}_+} \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \longrightarrow \sum_{n\in\mathbb{Z}_+} \begin{pmatrix} A_{33}^{(n)} & -A_{32}^{(n)} & A_{31}^{(n)} \\ -A_{23}^{(n)} & A_{22}^{(n)} & -A_{21}^{(n)} \\ A_{13}^{(n)} & -A_{12}^{(n)} & A_{11}^{(n)} \end{pmatrix} \tag{3.5.37}$$

$$K = \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} A_{11}^{(n)}+A_{33}^{(n)} & A_{12}^{(n)}-A_{32}^{(n)} & A_{13}^{(n)}+A_{31}^{(n)} \\ A_{21}^{(n)}-A_{23}^{(n)} & 2A_{22}^{(n)} & A_{23}^{(n)}-A_{21}^{(n)} \\ A_{31}^{(n)}+A_{13}^{(n)} & A_{32}^{(n)}-A_{12}^{(n)} & A_{11}^{(n)}+A_{33}^{(n)} \end{pmatrix}$$
$$= \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} (\widehat{A}_{11}^{(n)})_+ & (\widehat{A}_{12}^{(n)})_- & (\widehat{A}_{13}^{(n)})_+ \\ (\widehat{A}_{21}^{(n)})_- & (\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_- \\ (\widehat{A}_{31}^{(n)})_+ & (\widehat{A}_{32}^{(n)})_- & (\widehat{A}_{33}^{(n)})_+ \end{pmatrix}$$

and hence $K \in \mathfrak{so}_1^{(2)}(3)$.

$$P = \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} A_{11}^{(n)}-A_{33}^{(n)} & A_{12}^{(n)}+A_{32}^{(n)} & A_{13}^{(n)}-A_{31}^{(n)} \\ A_{21}^{(n)}+A_{23}^{(n)} & 0 & A_{23}^{(n)}+A_{21}^{(n)} \\ A_{31}^{(n)}-A_{13}^{(n)} & A_{32}^{(n)}+A_{12}^{(n)} & A_{33}^{(n)}-A_{11}^{(n)} \end{pmatrix}$$
$$= \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} (\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_- \\ (\widehat{A}_{21}^{(n)})_+ & 0 & (\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_- & (\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_- \end{pmatrix}.$$

Thus $K+iP$ is

$$\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} (\widehat{A}_{11}^{(n)})_+ + i(\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_- + i(\widehat{A}_{12}^{(n)})_+ & (\widehat{A}_{13}^{(n)})_+ + i(\widehat{A}_{13}^{(n)})_- \\ (\widehat{A}_{21}^{(n)})_- + i(\widehat{A}_{21}^{(n)})_+ & (\widehat{A}_{22}^{(n)})_+ & (\widehat{A}_{23}^{(n)})_- + i(\widehat{A}_{23}^{(n)})_+ \\ (\widehat{A}_{31}^{(n)})_+ + i(\widehat{A}_{31}^{(n)})_- & (\widehat{A}_{32}^{(n)})_- + i(\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_+ + i(\widehat{A}_{33}^{(n)})_-. \end{pmatrix}$$

Hence the non-compact real form is $K+iP\oplus\mathbb{R}ic\oplus\mathbb{R}id \in \mathfrak{sl}_1^{(2)}(3,\mathbb{R})$. Therefore the affine Kac-Moody symmetric spaces are:

$$SU^{(2)}(3)/SO_1^{(2)}(3), \quad SL_1^{(2)}(3,\mathbb{R})/SO_1^{(2)}(3). \tag{3.5.38}$$

Case II:

$$U(t) = U(t)^{-1} = I_3, \quad u=-1 \quad \xi = 0. \tag{3.5.39}$$

Under the automorphism (3.5.2) with (3.5.39) the matrix

$$\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} & A_{13}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} & A_{23}^{(n)} \\ A_{31}^{(n)} & A_{32}^{(n)} & A_{33}^{(n)} \end{pmatrix} \longrightarrow \sum_{n\in\mathbb{Z}_+}\begin{pmatrix} -A_{33}^{(n)} & A_{23}^{(n)} & -A_{13}^{(n)} \\ A_{32}^{(n)} & -A_{22}^{(n)} & A_{12}^{(n)} \\ -A_{31}^{(n)} & A_{21}^{(n)} & -A_{11}^{(n)} \end{pmatrix}. \tag{3.5.40}$$

$$K = \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} A_{11}^{(n)}-A_{33}^{(n)} & A_{12}^{(n)}+A_{23}^{(n)} & 0 \\ A_{21}^{(n)}+A_{32}^{(n)} & 0 & A_{23}^{(n)}+A_{12}^{(n)} \\ 0 & A_{32}^{(n)}+A_{21}^{(n)} & A_{33}^{(n)}-A_{11}^{(n)} \end{pmatrix}$$
$$= \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} (\widehat{A}_{11}^{(n)})_- & (\widehat{A}_{12}^{(n)})_+ & 0 \\ (\widehat{A}_{21}^{(n)})_+ & 0 & (\widehat{A}_{23}^{(n)})_+ \\ 0 & (\widehat{A}_{32}^{(n)})_+ & (\widehat{A}_{33}^{(n)})_- \end{pmatrix}$$

and hence $K \in so^{(2)}_{-1}(3)$.

$$P = \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} A^{(n)}_{11}+A^{(n)}_{33} & A^{(n)}_{12}-A^{(n)}_{23} & A^{(n)}_{13} \\ A^{(n)}_{21}-A^{(n)}_{32} & 2A^{(n)}_{22} & A^{(n)}_{23}-A^{(n)}_{12} \\ A^{(n)}_{31} & A^{(n)}_{32}-A^{(n)}_{21} & A^{(n)}_{33}+A^{(n)}_{11} \end{pmatrix}$$
$$= \frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} (\widehat{A}^{(n)}_{11})_+ & (\widehat{A}^{(n)}_{12})_- & A^{(n)}_{13} \\ (\widehat{A}^{(n)}_{21})_- & 2A^{(n)}_{22} & (\widehat{A}^{(n)}_{23})_- \\ A^{(n)}_{31} & (\widehat{A}^{(n)}_{32})_- & (\widehat{A}^{(n)}_{33})_+ \end{pmatrix}.$$

Thus $K+iP$ is

$$\frac{1}{2}\sum_{n\in\mathbb{Z}_+}\begin{pmatrix} (\widehat{A}^{(n)}_{11})_- + i(\widehat{A}^{(n)}_{11})_+ & (\widehat{A}^{(n)}_{12})_+ + i(\widehat{A}^{(n)}_{12})_- & iA^{(n)}_{13} \\ (\widehat{A}^{(n)}_{21})_+ + i(\widehat{A}^{(n)}_{21})_- & 2iA^{(n)}_{22} & (\widehat{A}^{(n)}_{23})_+ + i(\widehat{A}^{(n)}_{23})_- \\ iA^{(n)}_{31} & (\widehat{A}^{(n)}_{32})_+ + i(\widehat{A}^{(n)}_{32})_- & (\widehat{A}^{(n)}_{33})_- + i(\widehat{A}^{(n)}_{33})_+ \end{pmatrix}.$$

Hence the non-compact real form is $K+iP\oplus\mathbb{R}ic\oplus\mathbb{R}id \in \mathfrak{sl}^{(2)}_{-1}(3,\mathbb{R})$. Therefore the affine Kac-Moody symmetric spaces are:

$$SU^{(2)}(3)/SO^{(2)}_{-1}(3), \quad SL^{(2)}_{-1}(3,\mathbb{R})/SO^{(2)}_{-1}(3). \tag{3.5.41}$$

Having completed the explicit determination of all affine Kac-Moody symmetric spaces associated with $A^{(1)}_1, A^{(1)}_2$ and $A^{(2)}_2$ by algebraic method and identifying them with the corresponding Vogan diagrams, now we are in a position to determine all the affine Kac-Moody symmetric spaces. For this we have used root systems of complex Lie algebras and their Dynkin diagrams, diagram automorphisms [FSS96] and root systems associated with real forms of affine Kac- Moody algebras. We have included these in detail so that the readers have a clear idea about the method of determination of affine Kac-Moody symmetric spaces from Vogan diagrams of corresponding algebras (Table 3.1-3.11).

3.5.4 Diagram automorphism

The diagram automorphism for affine Lie algebra are as follows. For $A^{(1)}_n$, the automorphism group of the Dynkin diagram is the dihedral group D_{n+1} which is generated by reflection $s: i \longmapsto n+1-i (\mathrm{mod}\, n+1)$ and the rotation $r: i \longmapsto i+1 (\mathrm{mod}\, n+1)$ which is of order $r+1$. For $D^{(1)}_r$, the automorphism group is generated by the vector automorphism σ_v, the spinor automorphism σ_s and the conjugation γ. σ_v acts as $0 \longleftrightarrow 1$, $r \longleftrightarrow r-1$ and $i \longmapsto i$ else, and hence is of order 2. The map γ acts as $r \longleftrightarrow r-1$ and $i \longmapsto i$ else. If r is even, σ_s acts as $i \longmapsto r-i$ is of order 2 while for odd r the prescription $i \longmapsto r-i$ only holds for $2 \leq i \leq r-2$ and is supplemented by $0 \longmapsto r \longmapsto 1 \longmapsto r-1 \longmapsto 0$. For the untwisted algebra $\mathfrak{g} = B^{(1)}_r, C^{(1)}_r$ and for twisted algebra $\mathfrak{g} = B^{(2)}_r, C^{(2)}_r$, there is only a single non-trivial automorphism γ which is a reflection [FSS96].

3.5.5 Non-compact real forms of untwisted affine Kac-Moody algebras

Let $\hat{\mathfrak{g}}_{\mathbb{R}}$ be a real affine Kac-Moody Lie algebra and $\mathfrak{t}_0$ be the fixed subalgebra of a Cartan involution on $\hat{\mathfrak{g}}_{\mathbb{R}}$. Consider $\mathfrak{c}_o$ is the center of $\mathfrak{t}_0$, then simple roots of $\mathfrak{t}_o$ are obtained as follows. When the automorphism in the Vogan diagram is non trivial, we know that $\mathfrak{t}_o$ is semisimple. The simple roots for $\mathfrak{t}_o$ then include the compact imaginary simple roots and the average of the members of each 2-element orbit of simple roots. If the Vogan diagram has no painted imaginary root, there is no other simple root for $\mathfrak{t}_o$. Otherwise there is one other simple root for $\mathfrak{t}_o$, obtained by taking a minimal complex root containing the painted imaginary root in its expansion and averaging it over its 2-element orbit under the automorphism. When the automorphism is trivial, either $\dim \mathfrak{c}_0 = 1$, in this case the simple roots for $\mathfrak{t}_o$ are the compact simple roots for $\mathfrak{g}_o$, or else $\dim \mathfrak{c}_0 = 0$, in this case the simple roots for $\mathfrak{t}_o$ are the compact imaginary simple roots for $\mathfrak{g}_o$ and one other compact imaginary root. In latter case this other compact imaginary root is the unique smallest root containing the non-compact simple root twice in its expansion. We have discussed all these details below, for each real algebra separately.

- $\mathfrak{sl}_s^{(1)}(n,\mathbb{H}), n$ even ≥ 2
  Vogan diagram:
  A_{n-1}, non trivial automorphism,
  no imaginary simple roots
  $\mathfrak{t}_o = \mathfrak{sp}^{(1)}(n)$
  Simple roots for $\mathfrak{t}_o$:
  $e_{2n} - e_1,\ e_n - e_{n+1}$ and
  all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_{-1}^{(1)}(2n,\mathbb{R}), n \geq 3$
  Vogan diagram:
  A_{n-1}, non trivial automorphism,
  unique imaginary simple root $e_{2n} - e_1$
  $\mathfrak{t}_o = \mathfrak{su}^{(2)}(2n)$
  Simple roots for $\mathfrak{t}_o$:
  $\frac{1}{2}(e_{n-1} + e_n - e_{n+1} - e_{n+2})$ and
  all $\frac{1}{2}(e_i - e_{i+1} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n-1)$
- $\mathfrak{sl}_{-1}^{(1)}(2n+1,\mathbb{R}), n \geq 3$
  Vogan diagram:
  A_{n-1}, non trivial automorphism,
  no imaginary simple roots
  $\mathfrak{t}_o = \mathfrak{su}^{(2)}(2n+1)$

Simple roots for $\mathfrak{t}_o$:

$e_{2n+1} - e_1,\ \frac{1}{2}(e_n - e_{n+2})$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{2n+1-i} - e_{2n+2-i})$ for $1 \leq i \leq (n-1)$

- $\mathfrak{sl}_1^{(1)}(2n, \mathbb{R}), n \geq 4$

Vogan diagram:

A_{n-1}, non trivial automorphism,
two imaginary simple roots $e_{2n} - e_1, e_n - e_{n+1}$

$\mathfrak{t}_o = \mathfrak{so}^{(1)}(2n)$

Simple roots for $\mathfrak{t}_o$:

$\frac{1}{2}(e_{n-1} + e_n - e_{n+1} - e_{n+2}), \frac{1}{2}(e_{2n-1} + e_{2n} - e_1 - e_2)$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{2n+1-i} - e_{2n+2-i})$ for $1 \leq i \leq (n-1)$

- $\mathfrak{sl}_1^{(1)}(2n+1, \mathbb{R}), n \geq 4$

Vogan diagram:

A_{n-1}, non trivial automorphism,
one imaginary simple roots $e_{2n+1} - e_1$

$\mathfrak{t}_o = \mathfrak{so}^{(1)}(2n)$

Simple roots for $\mathfrak{t}_o$:

$\frac{1}{2}(e_n - e_{n+2}), \frac{1}{2}(e_{2n} + e_{2n+1} - e_1 - e_2)$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{2n+1-i} - e_{2n+2-i})$ for $1 \leq i \leq (n-1)$

- $\mathfrak{sl}_{rn}^{(1)}(2n, \mathbb{H}), n \geq 4$

Vogan diagram:

A_{n-1}, non trivial automorphism,
no imaginary simple roots

$\mathfrak{t}_o = \mathfrak{su}^{(1)}(n)$

Simple roots for $\mathfrak{t}_o$:

$\frac{1}{2}(e_n - e_{n+1} + e_{2n} - e_1)$ and
all $\frac{1}{2}(e_i - e_{i+1} + e_{n+i} - e_{n+1-i})$ for $1 \leq i \leq (n-1)$

- $\mathfrak{sl}_{rs}^{(1)}(n, \mathbb{H}), n \geq 4$

Vogan diagram:

A_{n-1}, non trivial automorphism,
no imaginary simple roots

$\mathfrak{t}_o = \mathfrak{so}^{(2)}(2n)$

Simple roots for $\mathfrak{t}_o$:

$\frac{1}{2}(e_{2n} - e_2),\ \frac{1}{2}(e_n - e_{n+2})$ and
all $\frac{1}{2}(e_{i+1} - e_{i+2} + e_{2n-i} - e_{2n+1-i})$ for $1 \leq i \leq (n-2)$

- $\mathfrak{su}_{-1}^{(1)}(p,q), p+q = 2n$

Vogan diagram:

A_{2n-1}, trivial automorphism,
unique imaginary simple root $e_{2n} - e_1$

$\mathfrak{t}_o = \mathfrak{su}(2n)$

Simple roots for $\mathfrak{t}_o$:

compact simple roots only

- $\mathfrak{su}_1^{(1)}(p,q), p+q=2n$

  Vogan diagram:

  A_{2n-1}, trivial automorphism,

  two imaginary simple roots $e_{2n}-e_1, e_p-e_{p+1}$

  $\mathfrak{t}_o = \mathfrak{su}(p)\oplus\mathfrak{su}(q)$

  Simple roots for $\mathfrak{t}_o$:

  compact simple roots only

- $\mathfrak{so}_{-1}^{(1)}(2,2n-1)$

  Vogan diagram:

  B_n, trivial automorphism,

  one imaginary simple root e_1-e_2

  $\mathfrak{t}_o = \mathfrak{so}(2n+3)$

  Simple roots for $\mathfrak{t}_o$:

  compact simple roots only

- $\mathfrak{so}^{(1)}(2p,2q+1),\ p+q=n$

  Vogan diagram:

  B_n, trivial automorphism,

  one imaginary simple root e_p-e_{p+1}

$$\mathfrak{t}_o = \begin{cases} \mathfrak{so}(4)\oplus\mathfrak{so}(2n-3), & \text{if } p=2 \\ \mathfrak{su}^{(1)}(4)\oplus\mathfrak{so}(2n-5), & \text{if } p=3 \\ \mathfrak{so}^{(1}(2n) & \text{if } q=0 \\ \mathfrak{so}^{(1}(2p)\oplus\mathfrak{so}(2q), & \text{if else} \end{cases}$$

  Simple roots for $\mathfrak{t}_o$:

  compact simple roots and

$$\begin{Bmatrix} e_{p-1}+e_p & \text{when } p\geq 3 \\ \text{no other} & \text{when } p=2 \end{Bmatrix}$$

- $\mathfrak{so}^{(1)}(1,2n)$

  Vogan diagram:

  B_n, non-trivial automorphism,

  no imaginary simple roots are painted

  $\mathfrak{t}_o = \mathfrak{so}^{(2)}(2n)$

  Simple roots for $\mathfrak{t}_o$:

  compact simple roots only and $-e_2$

- $\mathfrak{so}_1^{(1)}(2,2n-1)$

  Vogan diagram:

B_n, trivial automorphism,

two imaginary simple roots are painted $e_1 - e_2,\ -e_1 - e_2$

$\mathfrak{t}_o = \mathfrak{so}(2n+1)$

Simple roots for $\mathfrak{t}_o$:

compact simple roots only

- $\mathfrak{so}^{(1)}(2p+1, 2q),\ p+q = n$

  Vogan diagram:

  B_n, non-trivial automorphism,

  one imaginary simple root $e_p - e_{p+1}$

$$\mathfrak{t}_o = \begin{cases} \mathfrak{su}(3) \oplus \mathfrak{so}(2n-3), & \text{if } p = 2 \\ \mathfrak{so}^{(2)}(2n), & \text{if } q = 2 \\ \mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}(2q+1), & \text{if else} \end{cases}$$

  Simple roots for $\mathfrak{t}_o$:

  compact simple roots and $-e_2, e_p$

- $\mathfrak{sp}^{(1)}(p,q),\ p+q = 2n$ or $p+q = 2n-1$

  Vogan diagram:

  C_n, trivial automorphism,

  p^{th} simple root painted $e_p - e_{p+1}$,

$$\mathfrak{t}_o = \begin{cases} \mathfrak{su}^{(1)}(2) \oplus \mathfrak{sp}(q) & \text{if } p = 1, \forall q \\ \mathfrak{sp}^{(1)}(p) \oplus \mathfrak{sp}(q) & \text{if } p > 1, \forall q \end{cases}$$

  Simple roots for $\mathfrak{t}_o$: compact simple roots and $2e_p$

- $\mathfrak{sp}^{(1)}_{-1}(n, \mathbb{R})$

  Vogan diagram:

  C_n, trivial automorphism,

  n^{th} simple root painted $2e_n$,

  $\mathfrak{t}_o = \mathfrak{sp}(n)$

  Simple roots for $\mathfrak{t}_o$: compact simple roots only

- $\mathfrak{sp}^{(1)}_{1}(n, \mathbb{R})$

  Vogan diagram:

  C_n, trivial automorphism,

  n^{th} and affine simple roots are painted $2e_n, -2e_1$

  $\mathfrak{t}_o = \mathfrak{su}(n)$

  Simple roots for $\mathfrak{t}_o$: compact simple roots only

- $\mathfrak{sp}^{(1)}(2n-1, \mathbb{R})$

  Vogan diagram:

  C_n, non-trivial automorphism,

  no imaginary simple root

  $\mathfrak{t}_o = \mathfrak{su}^{(2)}(2n)$

Simple roots for $\mathfrak{t}_o$:

$(e_{2n-1}-e_1)$ and

all $\frac{1}{2}(e_i-e_{i+1}+e_{2n-1-i}-e_{2n-i})$ for $1 \le i \le (n-1)$

- $\mathfrak{sp}^{(1)}(n,\mathbb{H})$

Vogan diagram:

C_n, non-trivial automorphism,

no imaginary simple root

$\mathfrak{t}_o = \mathfrak{su}^{(2)}(2n)$

Simple roots for $\mathfrak{t}_o$:

$(e_n-e_{n+1}),(e_{2n}-e_1)$ and

all $\frac{1}{2}(e_i-e_{i+1}+e_{2n-i}-e_{2n+1-i})$ for $1 \le i \le (n-1)$

- $\mathfrak{sp}^{(1)}(2n,\mathbb{R})$

Vogan diagram:

C_n, non-trivial automorphism,

unique imaginary simple root e_n-e_{n+1} painted

$\mathfrak{t}_o = \mathfrak{su}^{(2)}(2n)$

Simple roots for $\mathfrak{t}_o$:

$\frac{1}{2}(e_{n-1}+e_n-e_{n+1}-e_{n+2})$ and

all $\frac{1}{2}(e_i-e_{i+1}+e_{2n-i}-e_{2n+1-i})$ for $1 \le i \le (n-1)$

- $\mathfrak{so}^{(1)}(2p,2q),\ p+q=n$

Vogan diagram:

D_{p+q}, trivial automorphism,

p^{th} simple root painted e_p-e_{p+1},

$$\mathfrak{t}_o = \begin{cases} \mathfrak{su}^{(1)}(4)\oplus\mathfrak{so}(2n-6), & \text{if } p=3 \\ \mathfrak{so}^{(1)}(2p)\oplus\mathfrak{so}(2q), & \text{if } p>3 \end{cases}$$

Simple roots for $\mathfrak{t}_o$:

compact simple roots and

$$\begin{Bmatrix} e_{p-1}+e_p & \text{when } p\ge 2 \\ \text{no other} & \text{when } p=1 \end{Bmatrix}$$

- $\mathfrak{so}^{*(1)}(2n)$

Vogan diagram:

D_n, trivial automorphism,

two imaginary simple roots $(e_1-e_2),(e_{n-1}-e_n)$ painted

$\mathfrak{t}_o = \mathfrak{su}(n)$

Simple roots for $\mathfrak{t}_o$: compact simple roots only

- $\mathfrak{so}^{(1)}_{-1}(2,2n-2)$

Vogan diagram:

D_n, trivial automorphism,

two imaginary simple roots $(e_1-e_2), -(e_1-e_2)$ painted

$\mathfrak{t}_\circ = \mathfrak{su}(2n-2)$

Simple roots for $\mathfrak{t}_\circ$: compact simple roots only

- $\mathfrak{so}^{(1)}_{\sigma_v}(1,2n-1)$

Vogan diagram:

D_n, non-trivial automorphism,

no imaginary simple root painted

$\mathfrak{t}_\circ = \mathfrak{so}^{(2)}(2n-2)$

Simple roots for $\mathfrak{t}_\circ$: $-e_2$, e_{n-1} and all (e_i-e_{i+1}) for $2 \le i \le (n-2)$

- $\mathfrak{so}^{(1)}_{\sigma_v}(2p+1,2q+1),\ p+q=n-1$

Vogan diagram:

D_n, non-trivial automorphism,

p^{th} simple root painted e_p-e_{p+1},

$$\mathfrak{t}_\circ = \begin{cases} \mathfrak{su}(3)\oplus\mathfrak{so}(2n-5), & \text{if } p=2 \\ \mathfrak{so}^{(2)}(2p)\oplus\mathfrak{so}^{(1)}(2q+1), & \text{if } p\ge 3 \end{cases}$$

Simple roots for $\mathfrak{t}_\circ$:

$-e_2$, e_{n-1}, e_p and

e_i-e_{i+1} for $2\le i\le p-1$ and $p+1\le i\le n-2$

- $\mathfrak{so}^{(1)}_{\gamma}(1,2n-1)$

Vogan diagram:

D_n, non-trivial automorphism,

no imaginary simple root painted

$\mathfrak{t}_\circ = \mathfrak{so}^{(1)}(2n-1)$

Simple roots for $\mathfrak{t}_\circ$:

$(e_1-e_2), -(e_1+e_2)$, e_{n-1} and

all (e_i-e_{i+1}) for $2\le i\le (n-2)$

- $\mathfrak{so}^{(1)}_{\gamma}(2p+1,2q+1),\ p+q=n-1$

Vogan diagram:

D_n, non-trivial automorphism,

p^{th} simple root painted e_p-e_{p+1},

$$\mathfrak{t}_\circ = \begin{cases} \mathfrak{so}^{(2)}(2n), & \text{if } p=1 \\ \mathfrak{sp}^{(1)}(2)\oplus\mathfrak{so}(2q+1), & \text{if } p=2 \\ \mathfrak{so}^{(1)}(2p+1)\oplus\mathfrak{so}(2q+1), & \text{if } p\ge 3 \end{cases}$$

Simple roots for $\mathfrak{t}_\circ$:

For $p=1$ $\quad e_p, -(e_1+e_2), e_{n-1}$,

all (e_i-e_{i+1}) for $p+1\le i\le (n-2)$

For $p\ne 1$ $\quad (e_1-e_2), -(e_1+e_2)$, e_{n-1}, e_p and

all (e_i-e_{i+1}) for $2\le i\le (p-1)$ and $p+1\le i\le (n-2)$

- $\mathfrak{so}_{\sigma_s}^{(1)}(1,2n-1), n$ even
  Vogan diagram:
  D_n, non-trivial automorphism,
  no imaginary simple root painted
  $\mathfrak{t}_o = \mathfrak{su}^{(2)}(n)$
  Simple roots for $\mathfrak{t}_o$:
  $\frac{1}{2}(e_1 - e_2 + e_{n-1} - e_n)$, $\frac{1}{2}(-e_1 - e_2 + e_{n-1} + e_n)$, $(e_{\frac{n}{2}} - e_{\frac{n}{2}+1})$ and all $\frac{1}{2}(e_i - e_{i+1} + e_{n-i} - e_{n+1-i})$ for $2 \leq i \leq (n-2)$
- $\mathfrak{so}_{\sigma_s}^{(1)}(1,2n-1), n$ even
  Vogan diagram:
  D_n, non-trivial automorphism,
  unique imaginary simple root $(e_{\frac{n}{2}} - e_{\frac{n}{2}+1})$ painted
  $\mathfrak{t}_o = \mathfrak{so}^{(1)}(n)$
  Simple roots for $\mathfrak{t}_o$:
  $\frac{1}{2}(e_1 - e_2 + e_{n-1} - e_n)$, $\frac{1}{2}(-e_1 - e_2 + e_{n-1} + e_n)$,
  $\frac{1}{2}(e_{\frac{n}{2}-1} + e_{\frac{n}{2}} - e_{\frac{n}{2}+1} - e_{\frac{n}{2}+2})$ and
  all $\frac{1}{2}(e_i - e_{i+1} + e_{n-i} - e_{n+1-i})$ for $2 \leq i \leq (n-2)$
- $\mathfrak{so}_{\sigma_s}^{(1)}(1,2n-1), n$odd
  Vogan diagram:
  D_n, non-trivial automorphism,
  no imaginary simple root painted
  $\mathfrak{t}_o = \mathfrak{so}^{(1)}(n)$
  Simple roots for $\mathfrak{t}_o$:
  $\frac{1}{2}(e_1 - e_2 + e_{n-1} - e_n)$, $\frac{1}{2}(-e_1 - e_2 + e_{n-1} + e_n)$, and
  all $\frac{1}{2}(e_i - e_{i+1} + e_{n-i} - e_{n+1-i})$ for $2 \leq i \leq (n-2)$

Table 3.1: Affine Kac-Moody symmetric spaces associated with $A^{(1)}_{2n-1}$

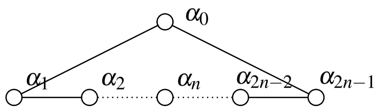
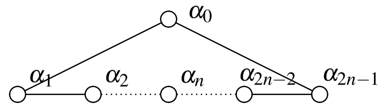
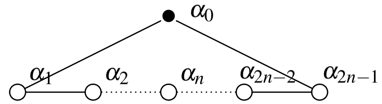
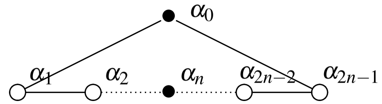
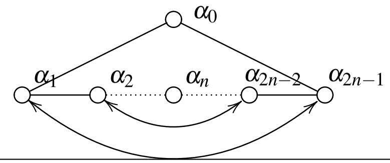
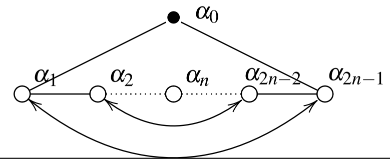
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{su}^{(1)}(2n)$		$\mathfrak{su}^{(1)}(2n)$		
	$\mathfrak{su}^{(1)}_{-1}(p,q)$, $p+q=2n$		$\mathfrak{su}(2n)$	$\frac{SU^{(1)}(p+q)}{SU(2n)}$	$\frac{SU^{(1)}_{-1}(p,q)}{SU(2n)}$
	$\mathfrak{su}^{(1)}_{1}(p,q)$, $p+q=2n$		$\mathfrak{su}(p)\oplus\mathfrak{su}(q)$	$\frac{SU^{(1)}(p+q)}{SU(p)\oplus SU(q)}$	$\frac{SU^{(1)}_{1}(p,q)}{SU(p)\oplus SU(q)}$
	$\mathfrak{sl}^{(1)}_{s}(n,\mathbb{H})$		$\mathfrak{sp}^{(1)}(2n)$	$\frac{SU^{(1)}(2n)}{SP^{(1)}(2n)}$	$\frac{SL^{(1)}_{s}(n,\mathbb{H})}{SP^{(1)}(2n)}$
	$\mathfrak{sl}^{(1)}_{-1}(2n,\mathbb{R})$, $n\geq 3$		$\mathfrak{su}^{(2)}(2n)$	$\frac{SU^{(1)}(2n)}{SU^{(2)}(2n)}$	$\frac{SL^{(1)}_{-1}(2n,\mathbb{R})}{SU^{(2)}(2n)}$

Table 3.2: Affine Kac-Moody symmetric spaces associated with $A_{2n-1}^{(1)}$ continued

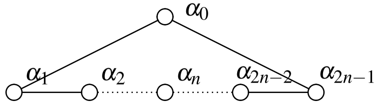
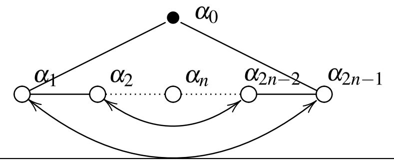
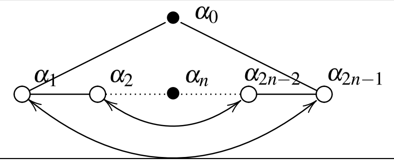
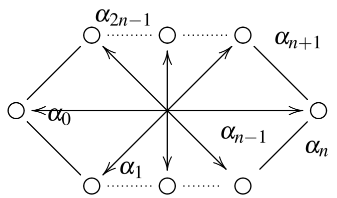
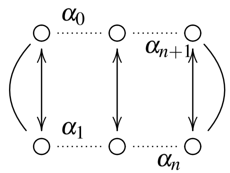
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{sl}_{-1}^{(1)}(2n,\mathbb{R})$, $n \geq 3$		$\mathfrak{su}^{(2)}(2n)$	$\frac{SU^{(1)}(2n)}{SU^{(2)}(2n)}$	$\frac{SL_{-1}^{(1)}(2n,\mathbb{R})}{SU^{(2)}(2n)}$
	$\mathfrak{sl}_{1}^{(1)}(2n,\mathbb{R})$, $n \geq 4$		$\mathfrak{so}^{(1)}(2n)$	$\frac{SU^{(1)}(2n)}{SO^{(1)}(2n)}$	$\frac{SL_{1}^{(1)}(2n,\mathbb{R})}{SO^{(1)}(2n)}$
	$\mathfrak{sl}_{r^n}^{(1)}(n,\mathbb{H})$		$\mathfrak{su}^{(1)}(n)$	$\frac{SU^{(1)}(2n)}{SU^{(1)}(n)}$	$\frac{SL_{r^n}^{(1)}(n,\mathbb{H})}{SU^{(1)}(n)}$
	$\mathfrak{sl}_{rs}^{(1)}(n,\mathbb{H})$		$\mathfrak{so}^{(2)}(2n)$	$\frac{SU^{(1)}(2n)}{SO^{(2)}(2n)}$	$\frac{SL_{rs}^{(1)}(n,\mathbb{H})}{SO^{(2)}(2n)}$

Table 3.3: Affine Kac-Moody symmetric spaces associated with $A_{2n}^{(1)}$

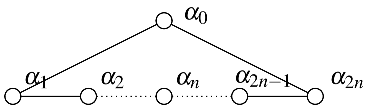
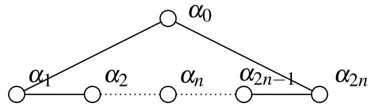
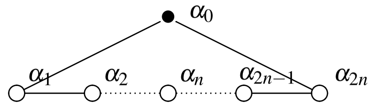
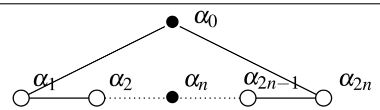
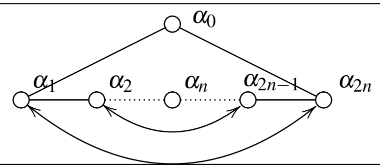
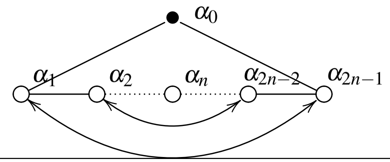
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{su}^{(1)}(2n+1)$		$\mathfrak{su}^{(1)}(2n+1)$		
	$\mathfrak{su}_{-1}^{(1)}(p,q)$, $p+q=2n+1$		$\mathfrak{su}(2n+1)$	$\frac{SU^{(1)}(p+q)}{SU(2n+1)}$	$\frac{SU_{-1}^{(1)}(p,q)}{SU(2n+1)}$
	$\mathfrak{su}_{1}^{(1)}(p,q)$, $p+q=2n+1$		$\mathfrak{su}(p)\oplus\mathfrak{su}(q)$	$\frac{SU^{(1)}(p+q)}{SU(p)\oplus SU(q)}$	$\frac{SU_{1}^{(1)}(p,q)}{SU(p)\oplus SU(q)}$
	$\mathfrak{sl}_{-1}^{(1)}(2n+1,\mathbb{R})$		$\mathfrak{su}^{(2)}(2n+1)$	$\frac{SU^{(1)}(2n+1)}{SU^{(2)}(2n+1)}$	$\frac{SL_{-1}^{(1)}(2n+1,\mathbb{R})}{SU^{(2)}(2n+1)}$
	$\mathfrak{sl}_{1}^{(1)}(2n+,\mathbb{R})$, $n\geq 3$		$\mathfrak{so}^{(1)}(2n)$	$\frac{SU^{(1)}(2n+1)}{SO^{(1)}(2n)}$	$\frac{SL_{1}^{(1)}(2n+1,\mathbb{R})}{SO^{(1)}(2n)}$

Table 3.4: Affine Kac-Moody symmetric spaces associated with $B_n^{(1)}$

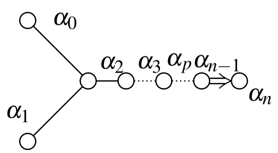
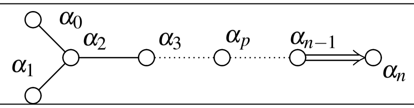
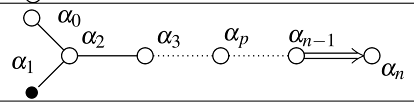
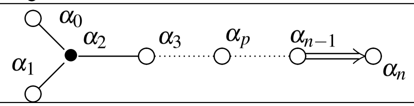
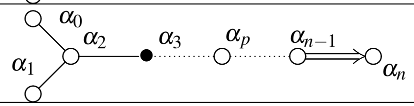
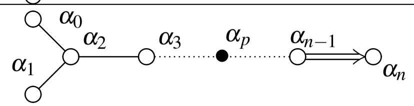
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{so}^{(1)}(2n+1)$		$\mathfrak{so}^{(1)}(2n+1)$		
	$\mathfrak{so}_{-1}^{(1)}(2,2n-1)$		$\mathfrak{so}(2n+1)$	$\frac{SO^{(1)}(2n+1)}{SO(2n+1)}$	$\frac{SO_{-1}^{(1)}(2,2n-1)}{SO(2n+1)}$
	$\mathfrak{so}^{(1)}(4,2n-3)$		$\mathfrak{so}(4)\oplus\mathfrak{so}(2n-3)$	$\frac{SO^{(1)}(2n+1)}{SO(4)\oplus SO(2n-3)}$	$\frac{SO^{(1)}(4,2n-3)}{SO(4)\oplus SO(2n-3)}$
	$\mathfrak{so}^{(1)}(6,2n-5)$		$\mathfrak{su}^{(1)}(4)\oplus\mathfrak{so}(2n-5)$	$\frac{SO^{(1)}(2n+1)}{SU^{(1)}(4)\oplus SO(2n-5)}$	$\frac{SO^{(1)}(6,2n-5)}{SU^{(1)}(4)\oplus SO(2n-5)}$
	$\mathfrak{so}^{(1)}(2p,2q+1)$ $p+q=n$		$\mathfrak{so}^{(1)}(2p)\oplus\mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n+1)}{SO^{(1)}(2p)\oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p,2q+1)}{SO^{(1)}(2p)\oplus SO(2q+1)}$

Table 3.5: Affine Kac-Moody symmetric spaces associated with $B_n^{(1)}$ continued

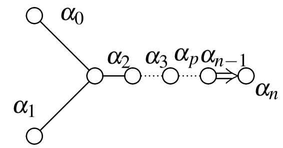
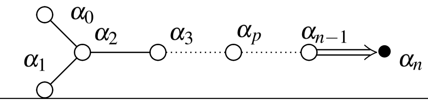
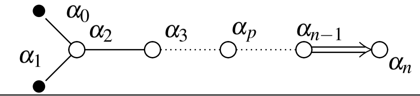
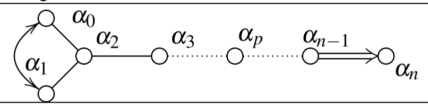
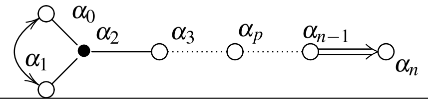
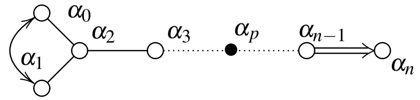
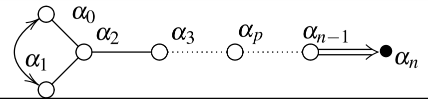
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{so}^{(1)}(2n,1)$		$\mathfrak{so}^{(1)}(2n)$	$\frac{SO^{(1)}(2n+1)}{SO^{(1)}(2n)}$	$\frac{SO^{(1)}(2n,1)}{SO^{(1)}(2n)}$
	$\mathfrak{so}_1^{(1)}(2,2n-1)$		$\mathfrak{so}(2n-1)$	$\frac{SO^{(1)}(2n+1)}{SO(2n-1)}$	$\frac{SO_1^{(1)}(2,2n-1)}{SO(2n-1)}$
	$\mathfrak{so}^{(1)}(1,2n)$		$\mathfrak{so}^{(2)}(2n)$	$\frac{SO^{(1)}(2n+1)}{SO^{(2)}(2n)}$	$\frac{SO^{(1)}(1,2n)}{SO^{(2)}(2n)}$
	$\mathfrak{so}^{(1)}(5,2n-4)$		$\mathfrak{su}(3)\oplus\mathfrak{so}(2n-3)$	$\frac{SO^{(1)}(2n+1)}{SU(3)\oplus SO(2n-3)}$	$\frac{SO^{(1)}(3,2n-2)}{SU(3)\oplus SO(2n-3)}$
	$\mathfrak{so}^{(1)}(2p+1,2q)$ $p+q=n$		$\mathfrak{so}^{(2)}(2p)\oplus\mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n+1)}{SO^{(2)}(2p)\oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p+1,2q)}{SO^{(2)}(2p)\oplus SO(2q+1)}$
	$\mathfrak{so}^{(1)}(2n-3,4)$		$\mathfrak{so}^{(2)}(2n)$	$\frac{SO^{(1)}(2n+1)}{SO^{(2)}(2n)}$	$\frac{SO^{(1)}(2n-3,4)}{SO^{(2)}(2n)}$

Table 3.6: Affine Kac-Moody symmetric spaces associated with $C_{2n-1}^{(1)}$

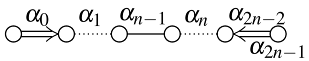
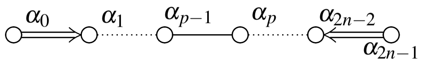
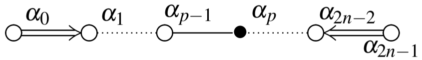
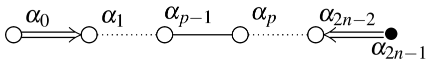
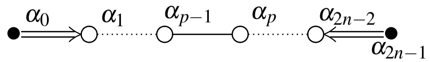
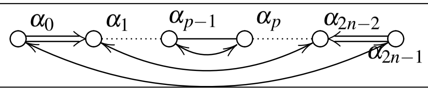
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{sp}^{(1)}(2n-1)$		$\mathfrak{sp}^{(1)}(2n-1)$		
	$\mathfrak{sp}^{(1)}(p,q)$, $p+q=2n-1$		$\mathfrak{sp}^{(1)}(p)\oplus\mathfrak{sp}(q)$	$\frac{SP^{(1)}(p+q)}{SP^{(1)}(p)\oplus SP(q)}$	$\frac{SP^{(1)}(p,q)}{SP^{(1)}(p)\oplus SP(q)}$
	$\mathfrak{sp}_{-1}^{(1)}(2n-1,\mathbb{R})$		$\mathfrak{sp}(2n-1)$	$\frac{SP^{(1)}(2n-1)}{SP(2n-1)}$	$\frac{SP_{-1}^{(1)}(2n-1,\mathbb{R})}{SP(2n-1)}$
	$\mathfrak{sp}_{1}^{(1)}(2n-1,\mathbb{R})$		$\mathfrak{su}(2n-1)$	$\frac{SP^{(1)}(2n-1)}{SU(2n-1)}$	$\frac{SP_{1}^{(1)}(2n-1,\mathbb{R})}{SU(2n-1)}$
	$\mathfrak{sp}^{(1)}(2n-1,\mathbb{R})$		$\mathfrak{su}^{(2)}(2n-1)$	$\frac{SP^{(1)}(2n-1)}{SU^{(2)}(2n-1)}$	$\frac{SP^{(1)}(2n-1,\mathbb{R})}{SU^{(2)}(2n-1)}$

Table 3.7: Affine Kac-Moody symmetric spaces associated with $C_{2n}^{(1)}$

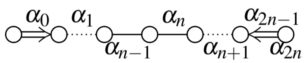
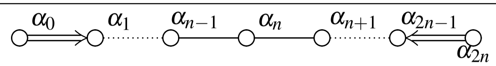
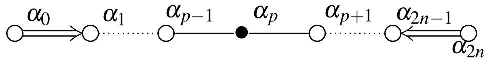
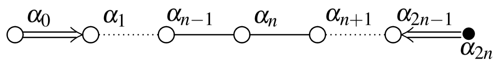
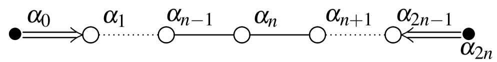
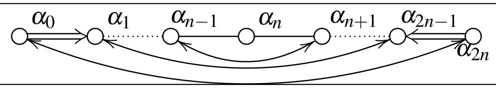
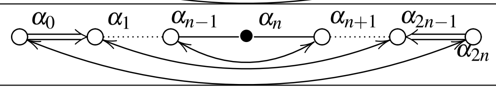
Dynkin Diagram	Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
	$\mathfrak{sp}^{(1)}(2n)$		$\mathfrak{sp}^{(1)}(2n)$		
	$\mathfrak{sp}^{(1)}(p,q)$, $p+q=2n,\ p>1$		$\mathfrak{sp}^{(1)}(p)\oplus\mathfrak{sp}(q)$	$\frac{SP^{(1)}(p+q)}{SP^{(1)}(p)\oplus SP(q)}$	$\frac{SP^{(1)}(p,q)}{SP^{(1)}(p)\oplus SP(q)}$
	$\mathfrak{sp}_{-1}^{(1)}(2n,\mathbb{R})$		$\mathfrak{sp}(2n)$	$\frac{SP^{(1)}(2n)}{SP(2n)}$	$\frac{SP_{-1}^{(1)}(2n,\mathbb{R})}{SP(2n)}$
	$\mathfrak{sp}_{1}^{(1)}(2n,\mathbb{R})$		$\mathfrak{su}(2n)$	$\frac{SP^{(1)}(2n)}{SU(2n)}$	$\frac{SP_{1}^{(1)}(2n,\mathbb{R})}{SU(2n)}$
	$\mathfrak{sp}^{(1)}(n,\mathbb{H})$		$\mathfrak{sp}^{(1)}(n)$	$\frac{SP^{(1)}(2n)}{SP^{(1)}(n)}$	$\frac{SP^{(1)}(n,\mathbb{H})}{SP^{(1)}(n)}$
	$\mathfrak{sp}^{(1)}(2n,\mathbb{R})$, $n\geq 3$		$\mathfrak{su}^{(2)}(2n)$	$\frac{SP^{(1)}(2n)}{SU^{(2)}(2n)}$	$\frac{SP^{(1)}(2n,\mathbb{R})}{SU^{(2)}(2n)}$

Table 3.8: Affine Kac-Moody symmetric spaces associated with $D_n^{(1)}$ for even n and $n > 4$

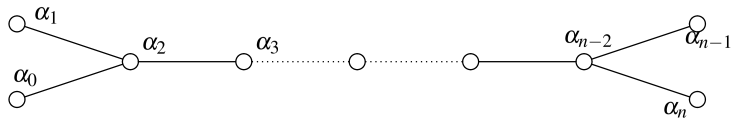
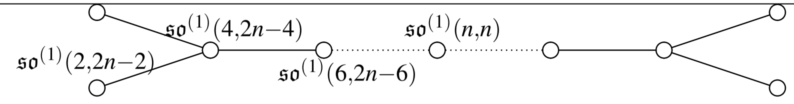
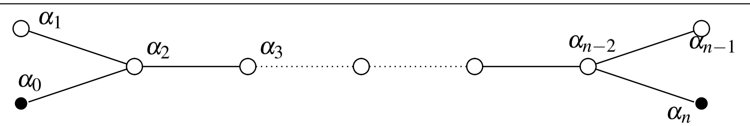
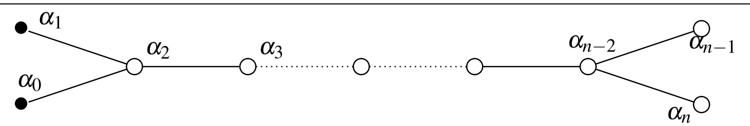
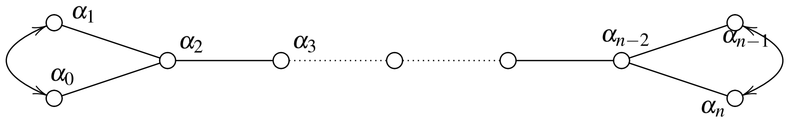
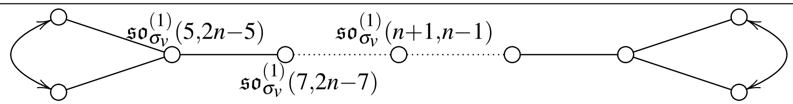
Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
$\mathfrak{so}^{(1)}(2n)$		$\mathfrak{so}^{(1)}(2n)$		
		$\mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}(2q)$	$\frac{SO^{(1)}(2p+2q)}{SO^{(1)}(2p)\oplus SO(2q)}$	$\frac{SO^{(1)}(2p,2q)}{SO^{(1)}(2p)\oplus SO(2q)}$
$\mathfrak{so}^{*(1)}(2n)$		$\mathfrak{su}(n)$	$\frac{SO^{(1)}(2n)}{SU(n)}$	$\frac{SO^{*(1)}(n)}{SU(n)}$
$\mathfrak{so}_{-1}^{(1)}(2,2n-2)$		$\mathfrak{so}(2n-2)$	$\frac{SO^{(1)}(2n)}{SO(2n-2)}$	$\frac{SO_{-1}^{(1)}(2,2n-2)}{SO(2n-2)}$
$\mathfrak{so}_{\sigma_v}^{(1)}(1,2n-1)$		$\mathfrak{sp}^{(2)}(2n-2)$	$\frac{SO^{(1)}(2n)}{SP^{(2)}(2n-2)}$	$\frac{SO_{\sigma_v}^{(1)}(1,2n-1)}{SP^{(2)}(2n-2)}$
		$\mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(2)}(2p)\oplus SO(2q+1)}$	$\frac{SO_{\sigma_v}^{(1)}(2p,2q+1)}{SO^{(2)}(2p)\oplus SO(2q+1)}$

Table 3.9: Affine Kac-Moody symmetric spaces associated with $D_n^{(1)}$ for even n and $n > 4$ continued

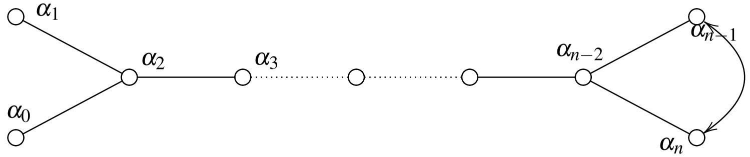
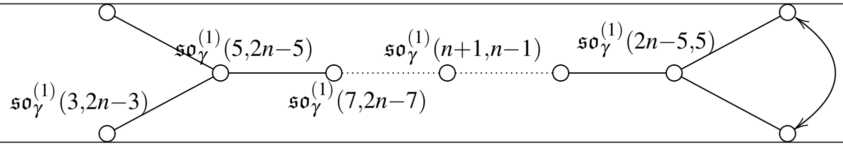
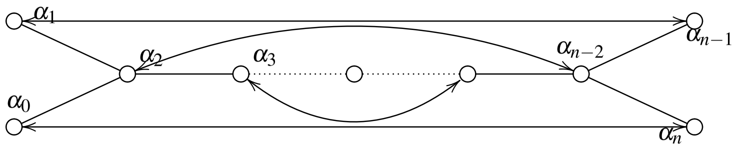
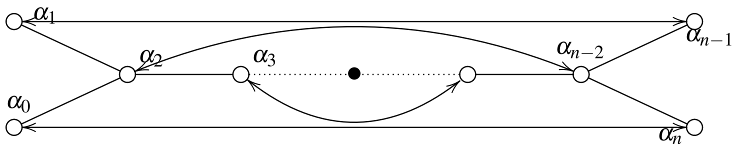
Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
$\mathfrak{so}_{\gamma}^{(1)}(1,2n-1)$		$\mathfrak{so}^{(1)}(2n-1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2n-1)}$	$\frac{SO_{\gamma}^{(1)}(1,2n-1)}{SO^{(1)}(2n-1)}$
		$\mathfrak{so}^{(1)}(2p+1) \oplus \mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2p+1) \oplus SO(2q+1)}$	$\frac{SO_{\gamma}^{(1)}(2p+1,2q+1}{SO^{(1)}(2p+1) \oplus SO(2q+1)}$
$\mathfrak{so}_{\sigma_s}^{(1)}(1,2n-1)$		$\mathfrak{su}^{(2)}(n)$	$\frac{SO^{(1)}(2n)}{SU^{(2)}(n)}$	$\frac{SO_{\sigma_s}^{(1)}(1,2n-1)}{SU^{(2)}(n)}$
$\mathfrak{so}_{\sigma_s}^{(1)}(n+1,n-1)$		$\mathfrak{so}^{(1)(n)}$	$\frac{SO^{(1)(2n)}}{SO^{(1)(n)}}$	$\frac{SO_{\sigma_s}^{(1)}(n+1,n-1)}{SO^{(1)(n)}}$

Table 3.10: Affine Kac-Moody symmetric spaces associated with $D_n^{(1)}$ for n odd and $n > 4$

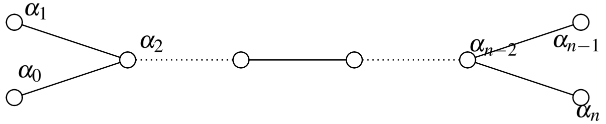
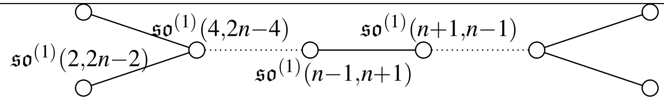
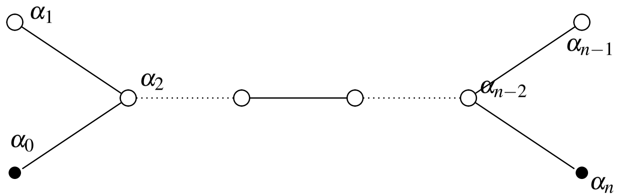
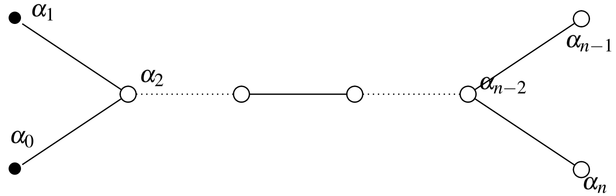
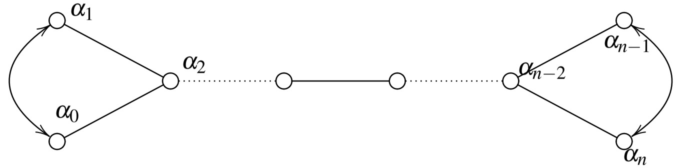
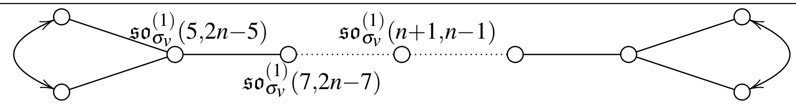
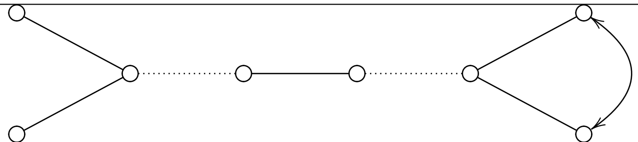
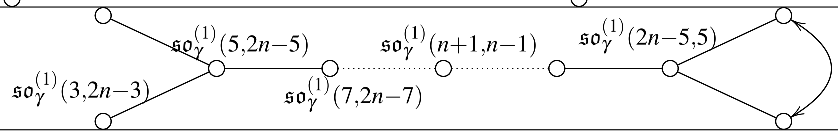
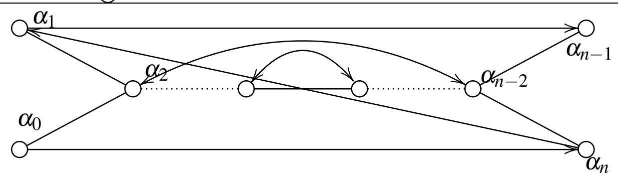
Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
$\mathfrak{so}^{(1)}(2n)$		$\mathfrak{so}^{(1)}(2n)$		
		$\mathfrak{so}^{(1)}(2p) \oplus \mathfrak{so}(2q)$	$\frac{SO^{(1)}(2p+2q)}{SO^{(1)}(2p)\oplus SO(2q)}$	$\frac{SO^{(1)}(2p,2q)}{SO^{(1)}(2p)\oplus SO(2q)}$
$\mathfrak{so}^{*(1)}(2n)$		$\mathfrak{su}(n)$	$\frac{SO^{(1)}(2n)}{SU(n)}$	$\frac{SO^{*(1)}(n)}{SU(n)}$
$\mathfrak{so}_{-1}^{(1)}(2,2n-2)$		$\mathfrak{so}(2n-2)$	$\frac{SO^{(1)}(2n)}{SO(2n-2)}$	$\frac{SO_{-1}^{(1)}(2,2n-2)}{SO(2n-2)}$

Table 3.11: Affine Kac-Moody symmetric spaces associated with $D_n^{(1)}$ for n odd and $n > 4$ continued

Real Forms	Vogan Diagram	Fixed Algebra	Compact affine Kac-Moody Symmetric spaces	Non-compact affine Kac-Moody Symmetric spaces
$\mathfrak{so}_{\sigma_v}^{(1)}(1,2n-1)$		$\mathfrak{sp}^{(2)}(2n-2)$	$\frac{SO^{(1)}(2n)}{SP^{(2)}(2n-2)}$	$\frac{SO_{\sigma_v}^{(1)}(1,2n-1)}{SP^{(2)}(2n-2)}$
		$\mathfrak{so}^{(2)}(2p) \oplus \mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(2)}(2p)\oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p,2q+1)}{SO^{(2)}(2p)\oplus SO(2q+1)}$
$\mathfrak{so}_{\gamma}^{(1)}(1,2n-1)$		$\mathfrak{so}^{(1)}(2n-1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2n-1)}$	$\frac{SO_{\gamma}^{(1)}(1,2n-1)}{SO^{(1)}(2n-1)}$
		$\mathfrak{so}^{(1)}(2p+1) \oplus \mathfrak{so}(2q+1)$	$\frac{SO^{(1)}(2n)}{SO^{(1)}(2p+1)\oplus SO(2q+1)}$	$\frac{SO^{(1)}(2p+1,2q+1)}{SO^{(1)}(2p+1)\oplus SO(2q+1)}$
$\mathfrak{so}_{\sigma_s}^{(1)}(1,2n-1)$		$\mathfrak{so}^{(2)}(n-1)$	$\frac{SO_{\sigma_s}^{(1)}(1,2n-1)}{SO^{(2)}(n-1)}$	$\frac{SO_{\sigma_s}^{(1)}(1,2n-1)}{SO^{(2)}(n-1)}$

3.6 Conclusion

In this chapter typically we have computed affine Kac-Moody symmetric spaces associated with the untwisted Kac-Moody algebras of classical types. It is interesting to determine explicitly the affine Kac-moody symmetric spaces associated with affine twisted Kac-moody algebras and exceptional algebras. But this requires a greater effort. However such studies are currently in progress. Lastly we hope through this chapter readers could get some ideas about algebraic structure associated with such spaces which shall be helpful to them for using such types of spaces in different physical problems as already done in case of symmetric spaces associated with simple Lie algebras.

3.7 Appendix

Notations used:

- $\mathfrak{gl}(n,\mathbb{C}),(\mathfrak{gl}(n,\mathbb{R}))$: {all $n\times n$ complex (real matrices)}
- $\mathfrak{sl}(n,\mathbb{C}),(\mathfrak{sl}(n,\mathbb{R}))$: {all $n\times n$ complex (real matrices) of trace zero}
- $\mathfrak{sl}_1^{(1)}(n,\mathbb{R})$: $\mathfrak{sl}(n,\mathbb{R})\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{R}ic\oplus\mathbb{R}id$ with $u=1$
- $\mathfrak{sl}_{-1}^{(1)}(n,\mathbb{R})$: $\mathfrak{sl}(n,\mathbb{R})\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{R}ic\oplus\mathbb{R}id$ with $u=-1$
- $\mathfrak{so}(n):\{X\in gl(n,\mathbb{R})\mid X+X^*=0\}$
- $\mathfrak{so}_1^{(1)}(n)$: $\mathfrak{so}(n)\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{R}ic\oplus\mathbb{R}id$ with $u=1$
- $\mathfrak{so}_{-1}^{(1)}(n)$: $so(n)\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{R}ic\oplus\mathbb{R}id$ with $u=-1$
- $\mathfrak{sp}(n):\{X\in gl(n,\mathbb{H})\mid X+X^*=0\}$
- $\mathfrak{su}(n):\{X\in gl(n,\mathbb{C})\mid X+X^*=0 \text{ and } TrX=0\}$
- $\mathfrak{su}^{(1)}(n):\{X\in su(n)\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{R}ic\oplus\mathbb{R}id\mid X+X^*=0 \text{ and } TrX=0\}$
- $\mathfrak{su}^{(2)}(n):\sum_{p=0}^{1}\sum_{j\ mod\ 2=p} t^j\otimes\mathfrak{su}(n)_p^{(2)}\oplus\mathbb{R}ic\oplus\mathbb{R}id$ where $\mathfrak{su}(n)_0^{(2)}$ and $su(n)_1^{(2)}$ are the eigenspaces corresponding to eigenvalues 1 and $e^{\pi i}$ respectively. That is $a\in su(n)_p^{(2)}$ if $a\in\mathfrak{su}(n)$ and $\psi_\tau(a)=e^{\pi ip}$ where ψ_τ is the is the outer automorphism of $\mathfrak{su}(n)$.
- $\mathfrak{u}(p,q):\left\{\begin{pmatrix}Z_1 & Z_2\\ \overline{Z}_2^t & Z_3\end{pmatrix}\mid \begin{matrix}Z_1,Z_3 \text{ skew Hermitian of order } p \text{ and } q\\ \text{respectively, } Z_2 \text{ is arbitrary}\end{matrix}\right\}$
- $\mathfrak{su}(p,q):\left\{\begin{pmatrix}Z_1 & Z_2\\ \overline{Z}_2^t & Z_3\end{pmatrix}\mid \begin{matrix}Z_1 \text{ skew Hermitian of order } p\\ Z_3 \text{ skew Hermitian of order } q\\ TrZ_1+TrZ_3=0,\ Z_2 \text{ is arbitrary}\end{matrix}\right\}$
- $\mathfrak{su}_1^{(1)}(p,q):\left\{\begin{pmatrix}X_1 & X_2\\ \overline{X}_2^t & X_3\end{pmatrix}\mid \begin{matrix}X_i=Z_i\otimes\mathbb{C}[t,t^{-1}]\\ X_1 \text{ skew Hermitian of order } p\\ X_3 \text{ skew Hermitian of order } q\\ TrX_1+TrX_3=0\ X_2 \text{ is arbitrary}\end{matrix}\right\}$

$$X_i = Z_i \otimes \mathbb{C}[t,t^{-1}]$$

- $\mathfrak{su}^{(1)}_{-1}(p,q) : \left\{ \begin{pmatrix} X_1 & X_2 \\ -\overline{X}_2^t & X_3 \end{pmatrix} \mid \begin{array}{c} X_1 \text{ skew Hermitian of order } p \\ X_3 \text{ skew Hermitian of order } q \\ TrX_1 + TrX_3 = 0,\ X_2 \text{ is arbitrary} \end{array} \right\}$
- $\mathfrak{su}^*(2n) : \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z_1} \end{pmatrix} \mid \begin{array}{c} Z_1, Z_2\ n \times n \text{ complex matrix} \\ TrZ_1 + Tr\overline{Z_1} = 0 \end{array} \right\}$
- $\mathfrak{sp}(n,\mathbb{C}) : \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \overline{Z}_3 & -Z_1^t \end{pmatrix} \mid \begin{array}{c} Z_i\ n \times n \text{ complex matrix} \\ Z_2, Z_2 \text{ are symmetric} \end{array} \right\}$

Chapter 4

Embedding of hyperbolic Kac-Moody superalgebras

4.1 Kac-Moody superalgebras

In this section we give brief introduction to the Kac-Moody superalgebras and their Dynkin diagrams by reproducing some definitions and theorems without proof so that it will lead us to a better idea about the classification of the hyperbolic Kac-Moody superalgebra.

Let τ be a subset of $I = \{1, \ldots, r\}$.

Definition 4.1.1. *To a given generalized Cartan matrix A and subset τ, we associate a contragradient Lie superalgebra $G(A, \tau)$ called Kac-Moody superalgebra with $3r$ generators e_i, f_i, h_i and a $\mathbb{Z}_2$ gradation defined by $\deg e_i = \deg f_i = 0$ if $i \notin \tau$, $\deg e_i = \deg f_i = 1$ if $i \in \tau$ and $\deg h_i = 0 \ \forall i$. The generators satisfy the set of relations,*

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_i \tag{4.1.1}$$

$$[e_i, e_i] = 0,\ [f_i, f_i] = 0,\ [e_i, f_i] = 0 \quad \text{if } a_{ii} = 0 \tag{4.1.2}$$

and the Serre relations,

$$(ad\ e_i)^{1-\tilde{a}_{ij}} e_j, (ad\ f_i)^{1-\tilde{a}_{ij}} f_j = 0;\ \forall i \neq j \tag{4.1.3}$$

where the matrix $\tilde{A} = (\tilde{a}_{ij})$ is obtained from the Cartan matrix $A = (a_{ij})$ of $G(A, \tau)$ by replacing positive off diagonal entries by -1. Here $[.,.]$ brackets stands for the graded product defined by $[x, y] = -(-1)^{(\deg x)(\deg y)} [y, x]$ and satisfy

$$[x, [y, z]] = [[x, y], z] + (-1)^{(\deg x)(\deg y)} [y, [z, x]]$$

and ad denotes the adjoint action $(ad\ x) y = [x, y]$.

In case of Kac-Moody Lie algebra the matrices a_{ij} and $\tilde{a}_{ij}$ coincide and the equation (4.1.3) reduces to the standard Serre relation. However in case of superalgebras the description given by above Serre relation leads, in general, to a larger superalgebra under consideration. So it is necessary to write supplementary relations involving more than two generators, in order to quotient the larger superalgebra and to recover the original one. These supplementary conditions appear when we deal with odd root α_i' of zero length (i.e., $(\alpha_i, \alpha_i) = 0$). The supplementary conditions depend on the different types of vertices which appear in the Dynkin diagrams. For example, in case of $A(m,n)$, if α_i is an odd root than the supplementary condition, $[[[e_{i-1}, e_i], e_{i+1}], e_i] = 0$ is necessary. Similarly different types of relations hold good for different types of vertices, the details of which can be found in [Yam99].

For our study, we consider $A = (a_{ij})$ be a symmetrizable and indecomposable Cartan matrix. A matrix A is said to be a indecomposable if it can't be reduced to a block diagonal form and it is symmetrizable GCM if it can be expressed as $A = DG$, where D is a diagonal matrix and G is symmetric matrix. The entries of D and G are, in general, rational numbers. Now we can think of symmetric matrix $G = G_{ij}$ as on the root space and make the following identifications. $D_{ij} = \frac{2}{(\alpha_i, \alpha_i)} \delta_{ij}$ where α_i is even simple root, (i.e., $i \notin \tau$) or a non-degenerate odd root. $D_{ij} = \delta_{ij}$, if α_i is a degenerate odd root, (i.e., $i \in \tau$ and $(\alpha_i, \alpha_i) = 0$) and for this $\alpha_i, G_{ij} = (\alpha_i, \alpha_j)$. A Kac-Moody superalgebra can be written as $G = G_{\overline{0}} \oplus G_{\overline{1}}$, where $G_{\overline{0}}$ is the even part and $G_{\overline{1}}$ is the odd part. Let $H \subset G_{\overline{0}}$ be the subalgebra of G generated by the h_i is known as Cartan subalgebra. The superalgebra $G(A, \tau)$ can be decomposed as $G = \bigoplus_\alpha G_\alpha$, where $G_\alpha = \{x \in G | [h, x] = \alpha(h)x, h \in H\}$ is called the root space of root α. Now consider $\Phi = \{\alpha \in H^* | G_\alpha \neq 0\}$ is the set of roots of G, where H^* is the dual of Cartan subalgebra of H. A root α is called even if $G_\alpha \subset G_{\overline{0}}$ and it is called odd $G_\alpha \subset G_{\overline{1}}$. The set of even roots is denoted as $\Phi_{\overline{0}}$ and set of odd roots is denoted as $\Phi_{\overline{1}}$. Also if α is an even or bosonic root then $(\alpha, \alpha) \neq 0$ and 2α is not a root. The odd fermionic roots are further divided into isotropic(i.e.$(\alpha, \alpha) = 0$) and nonisotropic(i.e.$(\alpha, \alpha) \neq 0$ and 2α is a root) odd roots. G admits a Borel decomposition, i.e, $G = H^+ \bigoplus H \bigoplus H^-$ where H^+ and H^- are subalgebra such that $[H, H^+] \subset H^+$ and $[H, H^-] \subset H^-$ with $\dim H^+ = \dim H^-$. A root α is called positive if $G_\alpha \cap H^+ \neq \emptyset$ and a root α is called negative if $G_\alpha \cap H^- \neq \emptyset$. A root is called simple if it can't be written as sum of two positive roots. The set of simple roots is called the simple root system of G and is denoted by Π. The simple roots, α_j are linearly independent elements in H^* satisfy the relation $\alpha_j(h_i) = a_{ij}$. The Weyl group W of G is generated by the simple reflections r_i of H^* defined by $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \ \forall j$. The elements of the root system Φ that are W-conjugate to simple roots are called real roots, remaining are called imaginary roots. If Φ_{re} and Φ_{im} denotes the set of real and imaginary roots respectively, then $\Phi = \Phi_{re} + \Phi_{im}$.

To each simple root system of superalgebras a Dynkin diagram can be associated. A Dynkin diagram associated with a symmetric generalized Cartan matrix A is a graph on n vertices, with vertices i and j joined by $|\eta_{ij}|$ edges.

1. Each simple even root is represented as a white dot, each non-isotropic odd root($a_{ii} \neq 0$) is by a black dot and isotropic odd root ($a_{ii} = 0$) by a grey dot.
2. $$\begin{aligned} \eta_{ij} &= 2\frac{|a_{ij}|}{min(|a_{ii}|,|a_{jj}|)} && \text{if } a_{ii} \cdot a_{jj} \neq 0 \\ &= 2\frac{|a_{ij}|}{min_{a_{kk}\neq 0}|a_{kk}|} && \text{if } a_{ii} \neq 0 \text{ and } a_{jj} = 0 \\ &= |a_{ij}| && \text{if } a_{ii} = a_{jj} = 0. \end{aligned}$$
3. We add an arrow on the lines connecting i-th and j-th dots when $\eta_{ij} > 1$, pointing from i to j if $a_{ii} \cdot a_{jj} \neq 0$ and $|a_{ii}| > |a_{jj}|$ or if $a_{ii} = 0$ and $a_{jj} \neq 0$, $|a_{jj}| < 2$ and pointing from j to i if $a_{ii} = 0$ and $a_{jj} \neq 0$, $|a_{jj}| > 2$.

In Kac-Moody superalgebra G there are many inequivalent root systems (when they contain isotropic odd roots), upto a transformation of the Weyl group $W(G)$ of G. Hence the Weyl group $W(G)$ is extended by adding the following transformation called generalized Weyl transformation associated to isotropic odd roots of G. For $\alpha \in \Phi_{\bar{1}}$ define

$$\begin{aligned} w_\alpha(\beta) &= \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha && \text{if } \alpha,\alpha) \neq 0 \\ w_\alpha(\beta) &= \beta + \alpha && \text{if } (\alpha,\alpha) = 0 \text{ and } (\alpha,\beta) \neq 0 \\ w_\alpha(\beta) &= \beta && \text{if } (\alpha,\alpha) = 0 \text{ and } (\alpha,\beta) = 0 \\ w_\alpha(\alpha) &= \alpha. \end{aligned}$$

For each simple root systems of Kac-Moody superalgebra G there exists a particular simple root system which contains minimum number of odd root. This system is called a distinguished simple root system. But the Weyl transformation associated to an isotropic odd root α can't be lifted to an automorphism of the superalgebra. As W_α transforms even roots into odd ones and vice versa, $\mathbb{Z}_2$ gradation would not be respected.

4.2 Types of Kac-Moody superalgebras

Given a class of symmetrizable generalized Cartan matrix (GCM) and there associated algebras, we consider three types of superagebras.

1. Simple Lie superalgberas: These are finite dimensional Kac-Moody superalgebras.
2. Affine Kac-Moody superalgebras: These are the sets of infinite dimensional Lie superalgebras and are of two types:

a. Untwisted affine Kac-Moody superalgebras that correspond to identity automorphisms of corresponding simple Lie superalgebras;
b. Twisted affine Kac-Moody superalgebras corresponding to outer automorphisms of order 2 or 3 of the corresponding simple Lie superalgebras.

3. Indefinite Kac-Moody superalgebras: These are also the sets of infinite dimensional Lie superalgebras whose GCM are of indefinite type. A particular subclass of indefinite Kac-Moody superalgebras are hyperbolic Kac-Moody superalgebras.

Definition 4.2.1. *Let $G(A,\tau)$ be an indefinite Kac-Moody superalgebra with generalized Cartan matrix A and non trivial $\mathbb{Z}_2$-gradation τ corresponding to a connected Dynkin diagram. $G(A,\tau)$ is called a hyperbolic Kac-Moody (HKM) superalgebra if every leading principal submatrix of A decomposes into constituents of finite, or equivalently, if deleting a vertex of the Dynkin diagram, one gets the Dynkin diagrams of finite or affine type.*

The hyperbolic superalgebras are divided into the following classes:

1. *strictly hyperbolic if every leading principal submatrix of A decomposes into constituents of finite type;*
2. *purely hyperbolic if every leading principal submatrix of A decomposes into constituents of affine type;*
3. *hyperfinite if at least one leading principal submatrix of A decomposes into constituents of finite type;*
4. *hyperaffine if at least one leading principal submatrix of A decomposes into constituents of affine type.*

If the Cartan matrix A is symmetric, then the corresponding superalgebra $G(A,\tau)$ is called a *simplylaced* Kac-Moody superalgebra. Here we will consider only simplylaced superalgebras. The hyperbolic Kac-Moody superalgebra of rank two are infinite in number, but there is no simplylaced superalgebra of rank two with an even root or with a degenerate odd root only because, when $\tau = \{1\}$, the Cartan matrix $\begin{pmatrix} 0 & 1 \\ -k & 2 \end{pmatrix}$ with $k \in \mathbb{Z}_{>0}, k > 2$ is not symmetric. There are exactly 17 simplylaced hyperbolic Kac-Moody superalgebra in rank 3-6 [CCL10, FS05, TDP03] are given in Table 4.1.

All hyperbolic Kac-Moody superalgebras are classified and are also finite in number for rank > 2 with maximum possible rank being 6. In the non super case E_{10} is the only simplylaced hyperbolic Kac-Moody algebra of rank 10 which is also of highest rank. Unlike this there are three simplylaced hyperbolic Kac-Moody superalgebras of highest possible rank 6 namely $HA(0,4), HA(2,2)$ and $HD(4,1)$. However it is shown that $HA(0,4)$ and $HA(2,2)$ are embedded in $HD(4,1)$. So now it is believed that $HD(4,1)$ plays the same role in hyperbolic Kac-Moody superalgebras as that of E_{10} in hyperbolic

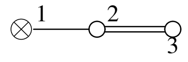
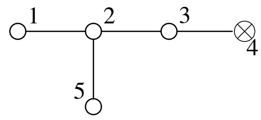
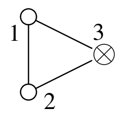
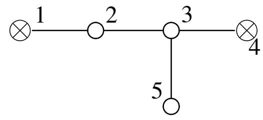
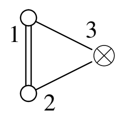
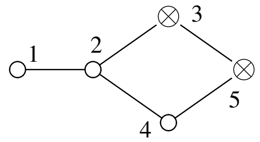
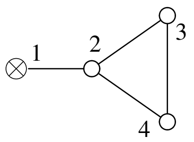
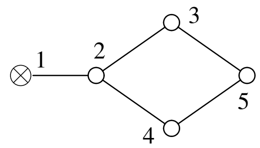
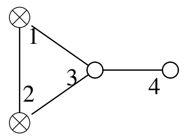
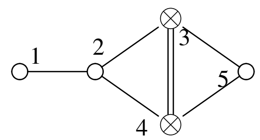
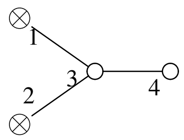
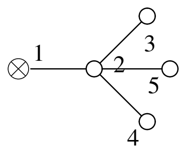
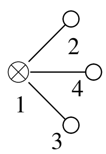
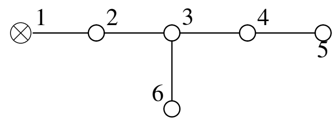
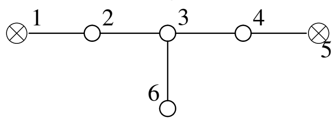
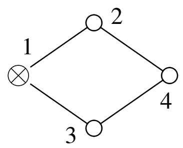
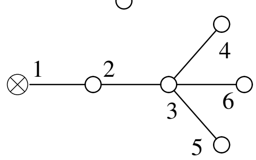
Name	Dynkin diagram	Name	Dynkin diagram
$HA(0,1)$		$HA(0,3)$	
HA_2		$HA(1,2)$	
$HA_1^{(1)}$		$HA(0,2)^{(1)}$	
$HA_2^{(1)}$		$HA_3^{(1)}$	
$HA(0,1)^{(1)}$		$HA(1,3)^{(2)}$	
$HA(0,2)$		$HD(3,1)$	
$HD(2,1)$		$HA(0,4)$	
		$HA(2,2)$	
HA_3		$HD(4,1)$	

Table 4.1: Simplylaced hyperbolic Kac-Moody superalgebras of rank 3, 4, 5, and 6

Kac-Moody algebra.

4.3 Main results

Our main aim is to prove the following:

Theorem 4.3.1. *Given any simplylaced hyperbolic Kac-Moody superalgebra (with a degenerate odd root in the Dynkin Diagram) G, there is a subalgebra(super) of rank 6 hyperbolic Kac-Moody superalgebra* $HD(4,1)$ *which is isomorphic to G.*

In order to prove this theorem we shall make use of the following Theorem 4.3.2 which is the supersymmetric version of the Theorem 3.1 of [FN04].

Theorem 4.3.2. *Let $G = G(A)$ be a Kac-Moody superalgebra associated with a $n \times n$ symmetric generalized Kac-matrix A. Let Φ denotes the root system of G with Cartan subalgebra H and Φ^+_{re} denotes the set of real positive roots of G. Consider $\beta_1, \beta_2, \ldots, \beta_k \in \Phi^+_{re}$, $k \leq n$ chosen such a way that $\beta_i - \beta_j \notin \Phi$, $\forall i \neq j$. Let G_{β_i} and $G_{-\beta_i}$ denote the one-dimensional root spaces corresponding to the positive real roots β_i and $-\beta_i$ respectively. $0 \neq e_{\beta_i} \in G_{\beta_i}$ and $0 \neq f_{\beta_i} \in G_{-\beta_i}$ are root vectors. Let $h_{\beta_i} = [e_{\beta_i}, f_{\beta_i}] \in H$ where $[.,.]$ denotes the Lie superbracket. Then $\{e_{\beta_i}, f_{\beta_i}, h_{\beta_i} \mid i = 1, 2, \ldots, k\}$ will generate a Kac-Moody subalgebra of rank k with Kac-matrix with $C = [c_{ij}] = [(\beta_i, \beta_j)]$. The subalgebra is denoted as $G(\beta_1, \beta_2, \ldots, \beta_k)$.*

In order to prove the hypothesis of the Theorem 4.3.2 we proceed as follows.

Proof. Let us denote $e_i = e_{\beta_i}$, $f_i = e_{-\beta_i}$. Now from construction we have $h_i = [e_i, f_i]$ and also $[e_i, f_i] \subseteq G_0$ where

$$G_0 = \{x \in G \mid [h, x] = 0, \forall h \in H\} = H. \tag{4.3.1}$$

So h_i are elements of Cartan subalgebra. Hence

$$[h_i, h_j] = 0, \ \ \forall\, i, j.$$

Again we have

$$[h_i, e_j] = \beta_j(h_i) e_j = a_{ij} e_j, \quad and \quad [h_i, f_j] = -\beta_j(h_i) f_j = -a_{ij} f_j \tag{4.3.2}$$

For $i \neq j$, $\ [e_i, f_j] \subseteq G_{\beta_i - \beta_j}$. But β_i are chosen such that $\beta_i - \beta_j \notin \Phi$ for $i \neq j$. $\beta_i - \beta_j$ can not be zero and as the superbracket of e_i, f_j must be zero for $i \neq j$. We have $[e_i, f_j] = \delta_{ij} h_i$. Finally we have to show, the following Serre relations also hold, i.e.,

$$\begin{aligned} (ad\ e_i)^{1-c_{ij}} e_j &= 0 \\ (ad\ f_i)^{1-c_{ij}} f_j &= 0; \quad \forall i \neq j, \end{aligned} \tag{4.3.3}$$

where $C = [(c_{ij})]$ is the matrix obtained from the generalized Cartan matrix $A = (a_{ij})$ replacing all positive off diagonal entries by -1. To show $(ade_i)^{1-c_{ji}} e_j = 0$ it is same thing to show that $(1 - c_{ji})\beta_i + \beta_j$ is not a root. Considering the β_i root string through β_j

$$\beta_j, \beta_j + \beta_i, \ldots, \beta_j + q\beta_i,$$

where $-q = 2\frac{(\beta_j, \beta_i)}{(\beta_i, \beta_i)} = c_{ij}$, since β_j are real roots of G and $\beta_i - \beta_j \notin \Phi$. Clearly, $\beta_j + (q+1)\beta_i = \beta_j + (1 - c_{ij})\beta_i$ is not in Φ. Similarly for $(ad\ f_i)^{1-c_{ij}} f_j = 0; \quad \forall \quad i \neq j$ as $-(1 - c_{ij})\beta_i - \beta_j \notin \Phi$. □

Now the following lemma is useful to verify the hypothesis of the above theorem:

Lemma 4.3.3. *If $\beta,\gamma \in \Phi_{re}^{+}$ satisfying $(\beta,\gamma) \leq 0$ then $\beta - \gamma \notin \Phi$. Here the bilinear form is supersymmetric: $(\alpha,\beta) = (-1)^{(\deg\alpha)(\deg\beta)}(\beta,\alpha)$.*

Proof. It is given that β, γ are real positive roots. To show their difference is not a root, we have to consider the following cases.

Case-1 (β, γ both are even roots).

Consider

$$\begin{aligned}(\beta-\gamma,\beta-\gamma) &= (\beta,\beta)+(\gamma,\gamma)-(\beta,\gamma)-(\gamma,\beta)\\ &= (\beta,\beta)+(\gamma,\gamma)-(\beta,\gamma)-(-1)^{\deg(\beta)deg(\gamma)}(\beta,\gamma)\\ &= (\beta,\beta)+(\gamma,\gamma)-2(\beta,\gamma) \geq 4\end{aligned}$$

as $deg(\beta) = deg(\gamma) = 0$ and $(\beta,\gamma) \leq 0$. So, $\beta-\gamma$ is not a imaginary root as it has positive norm. Also it is not a real root as norm is not 2. Hence it follows that $\beta-\gamma \notin \Phi$.

Case-II (β is even root and γ is odd degenerate root).

$(\beta-\gamma,\beta-\gamma) = (\beta,\beta)+(\gamma,\gamma)-(\beta,\gamma)-(\gamma,\beta) = 2-2(\beta,\gamma)$ as $\deg(\beta) = 1$ and $\deg(\gamma) = 0$. So $\beta-\gamma$ is not a imaginary root as it has positive norm. As $\beta-\gamma$ is an odd root so either $(\beta-\gamma,\beta-\gamma) = 0$ or $2(\beta-\gamma)$ should be a root. But as $(\beta-\gamma,\beta-\gamma) \geq 2$ is not a odd root of zero length. Again $(2(\beta-\gamma),2(\beta-\gamma)) = 4(\beta,\beta)+4(\gamma,\gamma)-4(\beta,\gamma)-4(\gamma,\beta) = 4-8(\beta,\gamma) \geq 4$ is not a odd black root also. Hence $\beta-\gamma \notin \Phi$.

Case-III (β, γ both are odd degenerate roots).

$(\beta-\gamma,\beta-\gamma) = (\beta,\beta)+(\gamma,\gamma)-(\beta,\gamma)-(\gamma,\beta) = -(\beta,\gamma)-(-1)^{(\deg\gamma)(\deg\beta)}(\beta,\gamma) = 0$. Now $\beta-\gamma$ is either an even root or is not a root. In the later case we are done. But in the former case $(\beta-\gamma,\beta-\gamma)$ should be 2. But then it is a contradiction as we have already shown $(\beta-\gamma,\beta-\gamma) = 0$. Hence $\beta-\gamma \notin \Phi$. □

As a consequence of the above lemma, if we want to check any collection of roots $\{\beta_i\}_{i=1}^{n}$ forms a subalgebra or not, simply we have to check whether $(\beta,\gamma) \leq 0$ or not, for $1 \leq i,j \leq n$.

Definition 4.3.4 ([Vis08]). *Suppose D denotes the Dynkin diagram of G and D' denotes the Dynkin diagram corresponding to the generalized Kac-matrix C then D' is called as root subdiagram of D. It is denoted as $D' \preceq D$.*

A root subdiagram needs not be a subdiagram of D. For proving Theorem 4.3.1 it is sufficient to prove the following proposition.

Proposition 4.3.5. *Every simplylaced connected hyperbolic Dynkin diagram (with degenerate odd root only) occurs as root subdiagram of rank 6 Dynkin diagram* $HD(4,1)$.

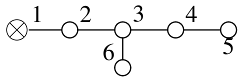
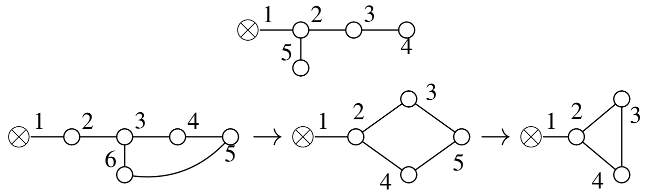
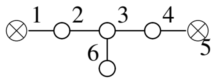
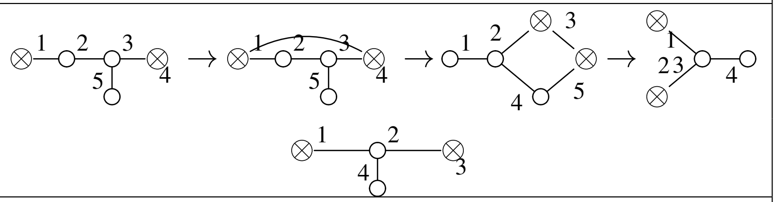
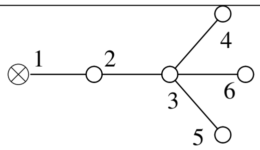
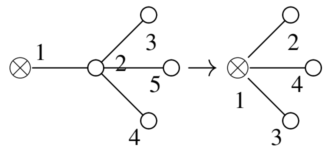
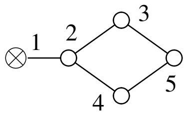
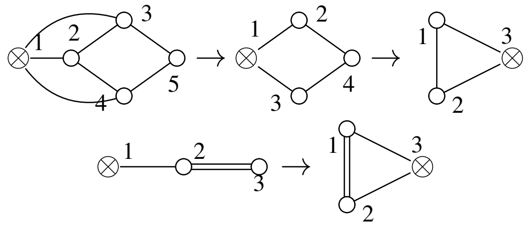
Diagram #1		→ →	
Diagram #2		→ →	
Diagram #3		→	
Diagram #4		→ →	

Table 4.2: Application of principles for proof of the main theorem

There are four general principles for constructing root subdiagrams, the details of which can be found in [Vis08] and which are given as follows.

Principle A: Let X be a simplylaced affine Dynkin diagram with simple roots $\alpha_0, \alpha_1, \ldots, \alpha_n$. Let $Y = HX$ be the hyperbolic extension of X and α_{-1} is the corresponding extra vertex of Y and which is connected with α_0 only. Then the root subdiagram is obtained by connecting the vertex α_{-1} with α_1, instead of α_0.

Principle B: Let X, Y, α_i $(-1 \leq i \leq n)$ as above. Choose a vertex $1 \leq p \leq n$ of X such that the root subdiagram of Y is obtained by connecting the vertex -1 to the chosen node p, and rest remains the same.

Principle B': We can also choose $F \subset \{1, 2, \ldots, n\}$, then the root subdiagram is obtained by connecting vertex -1 with all the vertices of F.

Principle C:(Shrinking)

Principle D:(Deletion) Given a Dynkin diagram Z with n vertices, deleting any set of vertices (and all incident edges)gives us a root subdiagram of Z.

From the list of Dynkin diagrams of simplylaced hyperbolic Kac-Moody algebras, we see that all Dynkin diagrams contain affine vertices. Principle A and B will work for hyperbolic extension of affine Dynkin diagrams only. However in our case there are some Dynkin diagrams which are hyperbolic without any affine vertex. Therefore we establish two principles which will work for our case.

Principle A^***:** Let X be a simplylaced Dynkin diagram with simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $Y = HX$ is the hyperbolic extension of X and α_{n+1} is the corresponding extra vertex of Y which is connected with some vertex α_i for some i. Then choose $a_1, a_2, \ldots, a_n \in \mathbb{C}$ with $a_i = 1$ such that

$$(\sum_{i=1}^{n} a_i\alpha_i, \alpha_j) = 0 \quad \forall j = 1,2,\ldots,n \text{ and } \; j \neq i.$$

Define $\beta_j = \alpha_j \forall j = 1,2,\ldots,n$ and $\beta_{n+1} = r_{\alpha_{n+1}}(\sum_{j=1}^{n} a_j\alpha_j + \alpha_j)$ where $r_{\alpha_{n+1}}$ is the simple reflection corresponding to α_{n+1}. Now we can observe that $\beta_{n+1} = \sum_{i=1}^{n} a_i\alpha_i + \alpha_i + 2\alpha_{n+1}$. Since β_{n+1} is the reflection of a real root, so $\beta_{n+1} \in \Phi_{re}$. Now the bilinear form is $(\beta_i, \beta_j) = (\alpha_i, \alpha_j)\,(1 \leq i, j \leq n)$. For $1 \leq j \leq n$ and $j \neq i$, we have

$$(\beta_{n+1}, \beta_j) = (\sum_{j=1}^{n} a_j\alpha_j + \alpha_i + 2\alpha_{n+1}, \beta_j) = (\alpha_i + 2\alpha_{n+1}, \beta_j)$$

$$\begin{aligned} &= 0 && \text{if } j = i \\ &= 0 && \text{if } j \neq i \text{ and } j \text{ is not connected to vertex } i \\ &= (\alpha_i, \alpha_j) && \text{if } j \neq i \text{ and } j \text{ is connected to vertex } i. \end{aligned}$$

So β_i satisfy the hypothesis of Theorem 4.3.2. The root subdiagram of Y is formed by $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1}$, where $\beta_j = \alpha_j \;\; \forall j = 1,2,\ldots,n$ and the extra vertex β_{n+1} is now connected with all the neighbours of vertex i in X. For example apply principle A^*

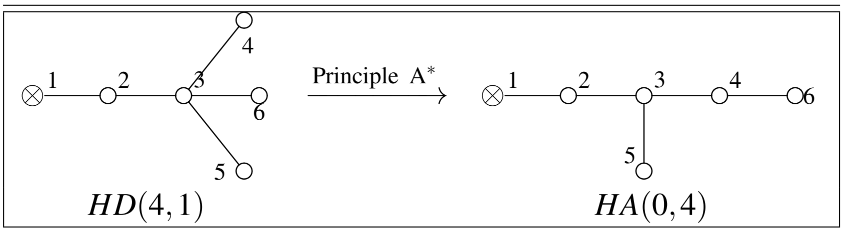


Table 4.3: $HA(0,4) \preceq HD(4,1)$

to $Y = HD(4,1)$ where $X = D(4,1)$, then β_i generate the following root subdiagram $HA(0,4)$. Thus $HA(0,4) \preceq Y = HD(4,1)$ as given in Table 4.3.

Principle B^***:** Let X, Y, and $\alpha_i\,(1 \leq i \leq n)$ as in Principle A^*. Then choose a vertex p of X

such that $(1 \leq p \leq n)$ and the vertex α_p is not connected with vertex α_{n+1}. We can define β_i as follows:
$\beta_j = \alpha_j \forall j = 1, 2, \ldots, n+1$ and $j \neq p$ and $\beta_p = \alpha_p + \sum_{i=1}^{n} a_i \alpha_i$. By this we have $\beta_j \in \Phi_{re}\,(1 \leq j \leq n+1)$ and the bilinear form satisfies $(\beta_i, \beta_j) = (\alpha_i, \alpha_j)\,(1 \leq i, j \leq n+1)$ and $i, j \neq p$. For $(1 \leq i \leq n+1)$ and $i \neq p$, we have

$$
\begin{aligned}
(\beta_p, \beta_i) &= (\alpha_p + \sum_{i=1}^{n} a_i \alpha_i, \beta_i) \\
&= -1 && \text{if } i = n+1 \\
&= (\alpha_p, \alpha_i) && \text{if } 1 \leq i \leq n) \text{ and } i \neq p.
\end{aligned}
$$

Thus, the root subdiagram of Y is formed by $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1}$ with an extra edge connecting the vertex $n+1$ to the chosen vertex p of X.

4.4 Proof of main result

All the above mentioned principles in section 4.3 can now be applied to prove the Proposition 4.3.5. In the proof we use the vertex number of the Dynkin diagrams, so readers are advised to see (Table 4.1) for numbering of vertices.

Proof. Here our claim is, all the simplylaced Dynkin diagram with degenerate odd roots only occur as root subdiagram of rank 6 Dynkin diagram $HD(4,1)$. In Table 4.2 first of all consider Diagram-#1. Let us apply Principle C to $HA(0,4)$ (i.e, add vertices 2 and 3 of the diagram), $HA(0,3)$ diagram is obtained. So $HA(0,3) \preceq HA(0,4)$. Now connect the vertex 6 of $HA(0,4)$ with vertex 5 (Principle B^*) and then add the vertices 2 and 3 (Principle C) of the new diagram obtained, we will get $HA_3^{(1)}$. By adding vertex 4 and 5 of $HA_3^{(1)}$ we will get $HA_2^{(1)}$. So, $HA_2^{(1)} \preceq HA_3^{(1)} \preceq HA(0,4)$.

Now in Diagram-#2 we can observe that $HA(1,2)$ is a root subdiagram of diagram $HA(2,2)$ via Principle C. Then connect the vertices 1 and 4 (Principle B^*) of the diagram $HA(1,2)$, that gives diagram $HA(0,2)^{(1)}$ and applying Principle C to the diagram obtained above we will get diagram $HA(0,1)^{(1)}$. Again diagram $HA(0,2)$ is obtained by adding vertex $2,3$ and 4 of the diagram $HA(2,2)$.

In Diagram-#3, $HD(3,1)$ and $HD(2,1)$ are obtained by using Principle C in the diagram $HD(4,1)$ and $HD(3,1)$ respectively.

Now consider Diagram-#4. Clearly by connecting vertices 3 and 4 with 1 in diagram $HA_3^{(1)}$ (Principle $B^{'}$) and deleting vertex 2 of the new diagram we will get HA_3 then adding 3 and 4 of the resulting diagram HA_2 can be obtained. Also by shrinking the vertex $3,4$ and 5 of $HA_3^{(1)}$ using Principle C we will get the diagram rank $HA(0,1)$. Then connect vertex 1 and 3 of $HA(0,1)$, we will get the diagram $HA_1^{(1)}$.

Now the rank 5 diagram $HA(1,3)^{(2)}$ can be embedded as follows:

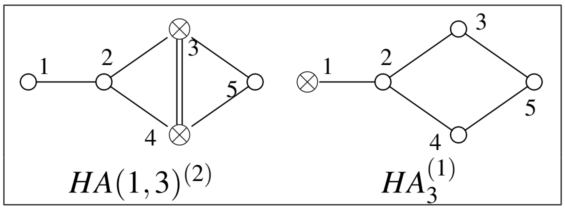


Table 4.4: $HA(1,3)^{(2)} \preceq HA_3^{(1)}$

The diagram $HA(1,3)^{(2)}$ can be obtained as root subdiagram of $HA(3)^{(1)}$ as above by choosing the vertices $\beta_1, \beta_2, \ldots, \beta_5$ as given below,

$$\beta_1 = \alpha_3, \quad \beta_2 = \alpha_2, \quad \beta_3 = \alpha_1, \quad \beta_5 = \alpha_5 \quad \textit{and}$$

$\beta_4 = \frac{8+2\sqrt{10}}{3}\alpha_1 + \frac{16+4\sqrt{10}}{3}\alpha_2 + \frac{8-\sqrt{10}}{6}\alpha_3 + \frac{2-\sqrt{10}}{2}\alpha_4 + \frac{2-\sqrt{10}}{3}\alpha_5.$
where α_i are simple roots of $HA_3^{(1)}$.

Finally we have one more diagram to be embedded, i.e., $HA(2,2)$.

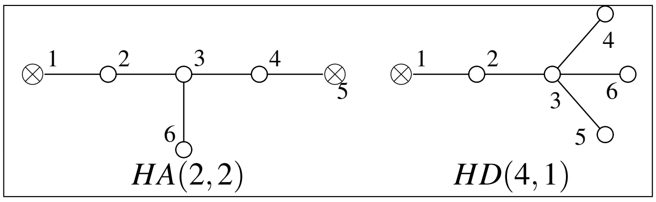


Table 4.5: $HA(2,2) \preceq HD(4,1)$

Let $\alpha_1, \alpha_2, \ldots, \alpha_6$ be the simple roots of $HD(4,1)$. Then $HA(2,2)$ can be obtained as the root subdiagram of $HD(4,1)$ by choosing $\beta_1, \beta_2, \ldots, \beta_6$ as follows:

$$\beta_i = \alpha_i \quad \textit{for} \quad i = 1,2,3,4. \quad \beta_6 = \alpha_5$$

$\beta_5 = \frac{1-\sqrt{3}i}{2}\alpha_1 + \frac{-1+\sqrt{3}i}{2}\alpha_3 + \frac{-3+\sqrt{3}i}{2}\alpha_4 + \frac{-1+\sqrt{3}i}{4}\alpha_5 + (1+\sqrt{3}i)\alpha_6.$
This completes the proof of the Proposition 4.3.5. □

4.5 Disconnected root subdiagrams

In the previous section we have shown that all connected simplylaced Dynkin diagrams, as root subdiagrams of $HD(4,1)$. But there are also some disconnected simply-laced root subdiagrams of $HD(4,1)$. Identifying all disconnected diagrams is a much harder task. However, to show how it happens we have proved the following proposition.

Proposition 4.5.1. *The rank* 6 *diagrams* $A(0,4) \oplus A_1$ *and* $D_4 \oplus A(0,1)$ *occurs as root subdiagrams of* $HD(4,1)$.

To prove this we need the concept of fundamental weights. Let us assume that $\{\Lambda_i\}_{i=1}^{6}$ are the fundamental weights of $HD(4,1)$, which satisfy the following:

$$2\frac{(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij} \qquad \text{for } (\alpha_j,\alpha_j) \neq 0$$

$$(\Lambda_i,\alpha_j) = \delta_{ij} \qquad \text{for } (\alpha_j,\alpha_j) = 0.$$

Here we list all Λ_i of the $HD(4,1)$ as linear combination of α_j:

$$\begin{aligned}
\Lambda_1 &= -\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6,\\
\Lambda_2 &= -\alpha_1,\\
\Lambda_3 &= -2\alpha_1 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\\
\Lambda_4 &= -\alpha_1 + \alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 + \frac{1}{2}\alpha_6,\\
\Lambda_5 &= -\alpha_1 + \alpha_3 + \frac{1}{2}\alpha_4 + \alpha_5 + \frac{1}{2}\alpha_6,\\
\Lambda_6 &= -\alpha_1 + \alpha_3 + \frac{1}{2}\alpha_4 + \frac{1}{2}\alpha_5 + \alpha_6.
\end{aligned}$$

Proof. Let $\alpha_i\,(1 \leq i \leq 6)$ be the simple roots of $HD(4,1)$. We have already proved that $HA(0,4) \preceq HD(4,1)$. Now deleting a vertex we will get $A(0,4)$. So $A(0,4) \preceq HD(4,1)$. Then we will get the simple roots $\{\beta_i\}_{i=1}^{5}$ of $A(0,4)$ such that,

$$\beta_i = \alpha_i,\ (1 \leq i \leq 4) \quad \textit{and} \quad \beta_5 = \alpha_1 - \alpha_3 - \alpha_4 - \frac{1}{2}\alpha_5 - \frac{1}{2}\alpha_6. \tag{4.5.1}$$

Now we are looking for a $\gamma \in \Phi_{re}^{+}$ such that

$$(\gamma,\beta_i) = 0 \quad i = 1,2,\ldots,5 \tag{4.5.2}$$

then $\{\beta_i\}_{i=1}^{5} \cup \{\gamma\}$ will generate $A(0,4) \oplus A_1$. From the equation (4.5.2) we can see that $\gamma \in \text{span}(\Lambda_5 - \Lambda_6)$. But $(\Lambda_5 - \Lambda_6, \Lambda_5 - \Lambda_6) = \frac{1}{2}(\alpha_5 - \alpha_6, \alpha_5 - \alpha_6) = 2$. Thus $\Lambda_5 - \Lambda_6$ is the required root.

From the Dynkin diagram $HD(4,1)$ it is clearly visible that D_4 is root subdiagram of $HD(4,1)$ by deleting vertices 1 and 2. So we have $D_4 \preceq HD(4,1)$. The simple roots $\{\beta_i\}_{i=1}^{4}$ of D_4 such that

$$\beta_1 = \alpha_3, \quad \beta_2 = \alpha_4, \quad \beta_3 = \alpha_5, \quad \beta_4 = \alpha_6.$$

Next we want to find a $\gamma \in \Phi_{re}^{+}$ such that

$$(\gamma,\beta_i) = 0 \quad \text{for} \quad i = 1,2,3,4. \tag{4.5.3}$$

This condition implies that $\gamma \in \text{span}(\Lambda_1,\Lambda_2)$. Let us take $\gamma_1 = \Lambda_1$ and $\gamma_2 = \Lambda_2$. It can be easily shown that $(\gamma_i,\gamma_i) = 0$ for $i = 1,2$. Also we have $(\gamma_1,\gamma_2) = -1$. Thus $\{\beta_i\}_{i=1}^{4} \cup \{\gamma_1,\gamma_2\}$ generates $D_4 \oplus A(0,1)$. This completes the proof. □

4.6 Conclusion

We have shown this result for hyperbolic Kac-Moody superalgebras which have been constructed from Lie superalgebras/ affine Kac-Moody superalgebras with distinguished bases. However one can extend such type of studies to hyperbolic Kac-Moody superalgebras with non distinguished bases. We hope that this chapter will account a small step towards understanding the structure of hyperbolic Kac-Moody superalgebras.

Chapter 5

Branching laws for infinite-dimensional Lie algebras

5.1 Introduction

Let $\mathfrak{g}$ be a Lie algebra over a complex field and $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$. If V is an irreducible $\mathfrak{g}$-module, V is no longer necessarily an irreducible $\mathfrak{g}'$-module. Branching law amounts to decompose V into irreducible $\mathfrak{g}'$-modules. In general such a decomposition doesn't necessarily exist. However if $\mathfrak{g}$ and $\mathfrak{g}'$ are semisimple (or reductive) Lie algebras and V is finite dimensional then $\mathfrak{g}$-module V decomposes as direct sum $V = \oplus m_j W_j$ where W_j is an irreducible $\mathfrak{g}'$-module and $m_j = \dim \mathrm{Hom}(W_j, V) \in \mathbb{N}_0$ is the multiplicity in V. In this situation, branching law amounts to identifying W_j in the above sum and also corresponding multiplicities m_j. In this nice setting also branching laws are not known completely, although there are specific formulas for the classical Lie algebra pairs $(\mathfrak{g}, \mathfrak{g}')$ for example $(\mathfrak{gl}(n), \mathfrak{gl}(n-1)$, $(\mathfrak{so}(n), \mathfrak{so}(n-1))$ and $(\mathfrak{sp}(n), \mathfrak{sp}(n-1))$ are known as classical branching laws.

Our aim is to study the branching laws for certain pairs $(\mathfrak{g}, \mathfrak{g}')$ of infinite dimensional Lie algebras, namely direct limit of finite dimensional reductive Lie algebras such as Lie algebra $\mathfrak{gl}(\infty)$ which can be viewed as algebra of infinite matrices with finitely many non-zero entries. One can also define $\mathfrak{gl}(\infty - 1)$ which play the same role in $\mathfrak{gl}(\infty)$ as $\mathfrak{gl}(n-1)$ plays in $\mathfrak{gl}(n)$. But it isn't priori clear whether such decomposition exists or not. So we have started with examples, to find the infinite dimensional counter part of classical branching laws. As a special case for pair $(\mathfrak{gl}(\infty), \mathfrak{gl}(\infty - 1))$ some branching laws have been found.

Before proceeding towards branching laws for different $\mathfrak{g}$-modules first we would like to give some preliminaries on highest weight representations and direct limit of Lie algebras, which are useful for rest of the chapter.

5.2 Highest weight representations

Let $\mathfrak{g}$ be a semisimple Lie algebra over a algebraically closed field of characteristic zero, $\mathfrak{h}$ is the Cartan subalgebra, Δ is the root system, Δ^+ is the positive root system, $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \subset \Delta^+$ be the simple root system and W denote the Weyl group. Let V be a finite dimensional $\mathfrak{g}$-module. With respect to $\mathfrak{h}$ we have a decomposition $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$. For $\lambda \in \mathfrak{h}^*$ whenever $V_\lambda \neq 0$ is called weight and V_λ is called weight space. Even if V is infinite dimensional the above definition for V_λ makes sense.

Proposition 5.2.1 ([Hum72]). *Let V be an (finite/infinite) $\mathfrak{g}$-module. Then*

1. *$\mathfrak{g}_\alpha$ maps V_λ into $V_{\lambda+\alpha}$ for $\lambda \in \mathfrak{h}^*, \alpha \in \Delta$.*
2. *The sum $V' = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda$ is direct and V' is an $\mathfrak{g}$-submodule of V.*
3. *If V is a finite dimensional then $V = V'$.*

Definition 5.2.2 ([Hum72]). *A highest weight vector (maximal vector) of weight λ in a $\mathfrak{g}$-module V is a nonzero vector $v^+ \in V_\lambda$ which is annihilated by all positive root vectors of $\mathfrak{g}$, i.e., $\mathfrak{g}_\alpha \cdot v^+ = 0$ for all $\alpha \in \Delta^+$ or simply for all $\alpha \in \Pi$.*

We know that if V is an $\mathfrak{g}$-module then it is also an $U(\mathfrak{g})$-module where $U(\mathfrak{g})$ is the universal enveloping algebra. Fix v^+, a maximal vector of weight λ and if $V = U(\mathfrak{g}) \cdot v^+$, i.e., if V is generated by the highest weight vector then V is called a highest weight module with highest weight λ and v^+ is called cyclic vector. Precisely, we have the following results.

Theorem 5.2.3 ([Hum72]). *Let V, W be two irreducible highest weight modules of highest weight λ then they are equivalent.*

Theorem 5.2.4 ([Hum72]). *Let V be a highest weight $\mathfrak{g}$-module, with maximal vector $v^+ \in V_\lambda$. Let $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$. Then,*

1. *V is spanned by the vectors $y_{\beta_1}^{i_1} \ldots y_{\beta_m}^{i_m} \cdot v^+$ for $i_j \in \mathbb{Z}$; in particular V is direct sum of its weight spaces with weights of the form $\mu = \lambda - \sum_{j=1}^{l} k_j \alpha_j$.*
2. *For each $\mu \in \mathfrak{h}^*, V_\mu$ is finite dimensional and* $\dim(V_\lambda) = 1$. *Moreover, each submodule of V is direct sum of its weight spaces.*
3. *V is an indecomposable $\mathfrak{g}$-module, with a unique maximal submodule and a corresponding unique irreducible quotient.*
4. *Every nonzero homomorphic image of V is also highest weight module of weight λ.*

Theorem 5.2.5 ([Hum72]). *For any $\lambda \in \mathfrak{h}^*$ there exists an irreducible highest weight module V_λ of weight λ.*

The linear functionals λ for which all $\lambda(h_i)$ are non-negative integers are called dominant integrals. Now we can classify all finite dimensional irreducible $\mathfrak{g}$-modules as follows.

Theorem 5.2.6 ([Hum72]). *Every finite dimensional irreducible $\mathfrak{g}$-module V is isomorphic to V_λ for some $\lambda \in \mathfrak{h}^*$. Moreover, $\lambda(h_i)$ is a non-negative integer $(1 \leq i \leq l)$ and any weight μ takes integer values on h_i.*

Theorem 5.2.7 ([Hum72]). *If $\lambda \in \mathfrak{h}^*$ is dominant integral then the irreducible $\mathfrak{g}$-module $V = V_\lambda$ is finite dimensional.*

So, apart from equivalence there is a one-one correspondence between the dominant algebraically integral linear functionals λ and the irreducible finite dimensional $\mathfrak{g}$-modules. Now we explicitly state here, the Weyl branching law which we further use in proving our result.

Theorem 5.2.8. *[Kna96a, Weyl Theorem] For the unitary Lie group $U(n)$, the irreducible representation with highest weight*

$$\lambda_1 e_1 + \cdots + \lambda_n e_n$$

decomposes with multiplicity 1 under $U(n-1)$ and the representations of $U(n-1)$ that appear are exactly those with highest weights $\sigma_1 e_1 + \cdots + \sigma_{n-1} e_{n-1}$ such that

$$\lambda_1 \geq \sigma_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \sigma_{n-1} \geq \lambda_n.$$

5.3 Direct limits of Lie algebras

Here we describe the definition of direct limit (inductive limit) of Lie group and Lie algebra [Nat94]. Let I be a directed set. Define a directed system $\{I, G_m, \tau_{m,n} : m, n \in I\}$, of finite dimensional Lie groups indexed by I to mean, G_m are finite dimensional Lie groups, $\tau_{m,n} : G_m \to G_n$ for $m \leq n$ are analytic group homomorphisms with $\tau_{n,l} \circ \tau_{m,n} = \tau_{m,l}$ for $m \leq n \leq l$ and $\tau_{m,m} = 1_{G_m}$.

The direct or inductive limit group $G = \varinjlim G_m$, consists of equivalence classes $[x_m]$ of sets x_m, with $\{x_m\} \in G_m$, and for some n, if $n \leq l$ then $x_l = \tau_{n,l}(x_n)$. Further, $\{x_m\} \sim \{x'_m\}$ if for some n, $n \leq l$ implies $x_l = x'_l$. We can thus define a homomorphism $\tau_m : G_m \to G$ defined by $\tau_m(x) = [x_n]$, where $x_n = \tau_{m,n}(x)$ for $m \leq n$, $x_n = 1_{G_n}$ otherwise.

The inductive limit topology on G is defined to be the strongest topology such that τ_m are continuous. In other words, a set U is open in G if and only if $\tau_m^{-1}(U)$ is open in G_m for each m.

If g_m is the Lie algebra of G_m with $d\tau_{m,n} : g_m \to g_n$, then we can similarly define the inductive limit Lie algebra $g = \varinjlim g_m$. Here $d\tau_{m,n}$ is the differential of $\tau_{m,n}$.

Example: Consider $G_n = GL(n,F)$ and the embedding maps $\tau_{n,n+1} : G_n \to G_{n+1}$ are defined by

$$\tau_{n,n+1}(X) = \begin{pmatrix} & & & 0 \\ & X & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5.3.1}$$

For $A \in \mathfrak{gl}(n,F)$, the Lie algebra of $GL(n,F)$, we define

$$\tau_{n,n+1}(A) = \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{5.3.2}$$

Define $GL(\infty) = \varinjlim GL(n,F)$ and $\mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n,F)$. Hence $\mathfrak{gl}(\infty)$ is the direct limit of the direct system $(\mathbb{N}, \mathfrak{gl}(n), \tau_{n,n+1})$, which is nothing but the set of $\infty \times \infty$ matrices with finite number of non-zero entries.

In the following section, we give branching law for the standard representation $\mathbb{C}^\infty$ (inductive limit of $\mathbb{C}^n$) of $\mathfrak{gl}(\infty)$ (inductive limit of $\mathfrak{gl}(n)$).

5.4 Branching law for $\mathbb{C}^\infty$

Before going to the infinite dimensional case, here we first give branching law for standard representation $\mathbb{C}^n$ of $\mathfrak{gl}(n)$ explicitly and then compare it with the Weyl branching law.

Theorem 5.4.1. *Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ be a complex Lie algebra. Then its standard representation, $(\pi, V = \mathbb{C}^n)$ is an irreducible $\mathfrak{g}$-module.*

Proof. Consider the action of $\mathfrak{g}$ on V, define by the map $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\pi(A)v = A \cdot v$ for $A \in \mathfrak{g}$ and $v \in \mathbb{C}^n$. With respect to the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices, we have the decomposition,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-. \tag{5.4.1}$$

$\mathfrak{g}_+ \oplus \mathfrak{h}$ is the Borel subalgebra, consist of upper triangular matrices. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. Clearly $\mathfrak{g}_+ e_1 = 0$ as $\mathfrak{g}_+$ consist of strictly upper triangular matrices. Now to find the highest weight λ consider the action of π on $\mathfrak{h}$. For $1 \le i \le n$ define the linear functionals ε_i on $\mathfrak{h}$ as

$$\varepsilon_i : \mathfrak{h} \to \mathbb{C} \text{ and } \varepsilon_i(h_j) = d_i \tag{5.4.2}$$

for $h_j \in \mathfrak{h}$. Here $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ is the dual basis for basis of $\mathfrak{h}$.

Now,

$$\pi(\mathfrak{h})e_1 = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = d_1 e_1 = \varepsilon_1(\mathfrak{h})e_1 \tag{5.4.3}$$

Since e_1 is annihilated by all positive weight vectors of $\mathfrak{g}$, it is the highest weight vector, and by (5.4.3) ε_1 is the highest weight. Again when one act $\mathfrak{g}_-$ on e_1, i.e., when we multiply lower triangular matrices with e_1 it will generate the whole vector space $\mathbb{C}^n$. This implies e_1 is a cyclic vector and hence $\mathbb{C}^n$ is an irreducible representation of $\mathfrak{g}$. □

Denote $\mathfrak{g}(n+1) = \mathfrak{gl}(n+1)$ and $\mathfrak{g}'(n+1) = \mathfrak{gl}(n)$.

Theorem 5.4.2. *With the embedding*

$$\begin{aligned} \mathfrak{g}'(n+1) &\hookrightarrow \mathfrak{g}(n+1) \\ A &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \end{aligned} \tag{5.4.4}$$

the irreducible $\mathfrak{g}(n+1)$*-module* $\mathbb{C}^{n+1}$ *is isomorphic to* $\mathbb{C} \oplus \mathbb{C}^n$ *as* $\mathfrak{g}'(n+1)$*-module.*

Proof. One can express $\mathbb{C}^{n+1}$ as a direct sum of $\mathbb{C}$ and $\mathbb{C}^n$, i.e.,

$$\mathbb{C}^{n+1} = \mathbb{C} \oplus \mathbb{C}^n$$

where $\mathbb{C}^{n+1} = \{(z_1, \dots, z_{n+1}) \mid z_i \in \mathbb{C}, i = 1, \dots, n+1\}$. Thus, we can think of $V = \mathbb{C}$ as one dimensional vector subspace of $\mathbb{C}^{n+1}$ generated by e_1 which is invariant under $\mathfrak{g}'(n+1)$, is the trivial representation. Similarly $W = \mathbb{C}^n$ is the vector subspace of $\mathbb{C}^{n+1}$ generated by $\{e_2, \dots, e_{n+1}\}$. Here W is invariant under $\mathfrak{g}'(n+1)$ as for $A \in \mathfrak{g}'(n+1)$ and for any $w \in W$ we have

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \begin{pmatrix} A \end{pmatrix}_{n\times n} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 0 \\ z_2 \\ \vdots \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ A \begin{pmatrix} z_2 \\ \vdots \\ z_{n+1} \end{pmatrix} \end{pmatrix} \in \mathbb{C}^{n+1},$$

is the standard representation. Thus,

$$\mathbb{C}^{n+1} \cong_{\mathfrak{g}'(n+1)} \mathbb{C} \oplus \mathbb{C}^n$$

□

$\mathbb{C}^{n+1}$ is the irreducible $\mathfrak{g}(n+1)$-module with highest weight $(1,0,\ldots,0) \in \mathbb{Z}_{n+1}$ and by Weyl branching law, $\mathbb{C}^{n+1}$ as irreducible $\mathfrak{g}'(n+1)$-module decomposes with highest weights $(0,\ldots,0) \in \mathbb{Z}_n$ and $(1,\ldots,0) \in \mathbb{Z}_n$. Precisely, $(0,\ldots,0)$ and $(1,\ldots,0)$ are highest weights of highest weight modules $\mathbb{C}$ and $\mathbb{C}^n$ respectively, which is in accordance to our result.

Theorem 5.4.2 says that the irreducible $\mathfrak{g}(n+1)$-module $\mathbb{C}^{n+1}$ can be expressed as direct sum of two $\mathfrak{g}'(n+1)$-module $\mathbb{C}$ and $\mathbb{C}^n$. Continuing in this manner, the irreducible $\mathfrak{g}(n+2)$-module $\mathbb{C}^{n+2}$ can be expressed as direct sum of two $\mathfrak{g}'(n+2)$-modules $\mathbb{C}$ and $\mathbb{C}^{n+1}$, i.e.,

$$\mathbb{C}^{n+2} \cong_{\mathfrak{g}'(n+2)} \mathbb{C} \oplus \mathbb{C}^{n+1}$$

and so on. Now consider the following chain

$$\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \cdots \subset \mathbb{C}^n \subset \cdots$$

and the homomorphisms which are given by embedding maps as follows.

$$\tau_n : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$$
$$(z_1,\ldots,z_n) \mapsto (z_1,\ldots,z_n,0).$$

So with respect to the direct system $(\mathbb{N}, \mathbb{C}^n, \tau_n)$, we have $\mathbb{C}^\infty := \varinjlim \mathbb{C}^n = \sqcup_{n\geq 1}\mathbb{C}^n$ is the inductive limit.

$\mathbb{C}^\infty$ is $\mathfrak{g}(\infty) = \mathfrak{gl}(\infty)$-module with action $\pi(A)v = A \cdot v$ for $A \in \mathfrak{g}(\infty)$ and $v \in \mathbb{C}^\infty$, here this multiplication makes sense as in $\mathfrak{g}(\infty)$ there are finite number of non-zero entries. Denote $\mathfrak{g}'(\infty) = \mathfrak{gl}(\infty - 1)$ which can be viewed as a set of matrices in the following form,

$$\mathfrak{g}'(\infty) = \left\{ A : A = \begin{pmatrix} 0 & 0 & \cdots & \cdots \\ 0 & a_{22} & a_{23} & \cdots \\ \vdots & \cdots & \cdots & \end{pmatrix} \right\} \subseteq \mathfrak{g}(\infty).$$

Theorem 5.4.3. *Suppose $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n$ is the standard representation of $\mathfrak{g}(\infty) = \varinjlim \mathfrak{g}(n)$. Then $\mathbb{C}^\infty \cong_{\mathfrak{g}'(\infty)} W_1 \oplus W_2$, where $W_1 = \varinjlim \mathbb{C}$ and $W_2 = \varinjlim \mathbb{C}^n = \sqcup_{n\geq 2}\mathbb{C}^n$.*

Proof. From Theorem 5.4.2, we infer that the irreducible $\mathfrak{g}(n+1)$-module $\mathbb{C}^{n+1}$ is isomorphic to $\mathbb{C}\oplus\mathbb{C}^n$ as $\mathfrak{g}'(n+1)$-module, i.e.,

$$\mathbb{C}^{n+1} \cong_{\mathfrak{g}'(n+1)} \mathbb{C}\oplus\mathbb{C}^n. \tag{5.4.5}$$

Further, for $\mathfrak{g}(n+2)$-module $\mathbb{C}^{n+2}$ we have

$$\mathbb{C}^{n+2} \cong_{\mathfrak{g}'(n+2)} \mathbb{C}\oplus\mathbb{C}^{n+1}. \tag{5.4.6}$$

To form the direct systems, define the embedding for $\mathfrak{g}'(n+1)$-module $\mathbb{C}^n$ in $\mathbb{C}^{n+1}$ into $\mathfrak{g}'(n+2)$-module $\mathbb{C}^{n+1}$ in $\mathbb{C}^{n+2}$ as

$$\begin{aligned} \mathfrak{g}'(n+1) &\hookrightarrow \mathfrak{g}'(n+2) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and the embedding map between $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$ as

$$\begin{aligned} \mathbb{C}^n &\hookrightarrow \mathbb{C}^{n+1} \\ (z_2,\ldots,z_{n+1}) &\mapsto (z_2,\ldots,z_{n+1},0). \end{aligned}$$

Thus like the above embedding, one can embed any $\mathfrak{g}'(n+1)$-module $\mathbb{C}^n$ into a $\mathfrak{g}'(n+k)$-module $\mathbb{C}^{n+k}$ as follows,

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_2 \\ \vdots \\ z_{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A\begin{pmatrix} z_2 \\ \vdots \\ z_{n+1} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.4.7}$$

Since $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n$ with the embedding

$$\begin{aligned} \mathbb{C}^{n+1} &\xrightarrow{\tau_{n+1}} \mathbb{C}^{n+2} \\ (z_1,\ldots,z_{n+1}) &\mapsto (z_1,\ldots,z_{n+1},0), \end{aligned}$$

we can decompose

$$\begin{aligned} \tau_{n+1}(z_1,\ldots,z_{n+1}) &= (z_1,0,\ldots,0)+(0,z_2\ldots,z_{n+1},0) \\ &= i_{n+1}(z_1,\ldots,z_{n+1})+j_{n+1}(z_1,\ldots,z_{n+1}). \end{aligned}$$

Set $W_1(n) := \mathbb{C}$ and $W_2(n) := \mathbb{C}^n$. Thus for the two direct systems $(\mathbb{N},\mathbb{C},i_n)$ and $(\mathbb{N},\mathbb{C}^n, j_n)$ we have the direct limits W_1 and W_2 respectively where $W_1 = \mathbb{C}$ and $W_2 = \sqcup_{n\geq 2}\mathbb{C}^n$. Also, W_2 is a $\mathfrak{g}(\infty-1)$-module in $\mathbb{C}^\infty$ which follows from (5.4.7) for large n and k positive. Since $W_1\cap W_2 = \{0\}$, hence the algebraic sum of W_1 and W_2 is exactly the direct sum. Now, the only thing is to show $\mathbb{C}^\infty = W_1\oplus W_2$. This follows from the fact that for any

$v \in \mathbb{C}^\infty$ implies $v \in \mathbb{C}^n$ for some n. □

5.5 Branching law for $S^k(\mathbb{C}^\infty)^*$

Consider $G = GL(n)$ is the Lie group of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$.

Theorem 5.5.1. *The finite dimensional G-module $V = S^k(\mathbb{C}^n)^*$ is an irreducible G-module.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and $\mathfrak{h}$ be the Cartan subalgebra. With respect to $\mathfrak{h}$ we have the following triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-.$$

Here $\mathfrak{g}_+, \mathfrak{g}_-$ are space of strictly upper and lower triangular matrices respectively and $\mathfrak{h}$ consist of diagonal matrices. Let $e_1, \ldots, e_n$ be the basis for $\mathbb{C}^n$ and $X_1, \ldots, X_n$ be the dual basis of $\mathbb{C}^n$ such that $X_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots n$ where δ_{ij} is the Kronecker delta function. Now the basis for k-th symmetric power of $(\mathbb{C}^n)^*$ is given by

$$\{X_{i_1} \cdot X_{i_2} \ldots X_{i_k} \mid 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n\}. \tag{5.5.1}$$

Action of G on $\mathrm{S}^k(\mathbb{C}^n)^*$ is defined as

$$g \cdot (X_{i_1} \ldots X_{i_k}) = (g \cdot X_{i_1}) \ldots (g \cdot X_{i_k}), \text{ for } g \in G.$$

Since $\mathrm{S}^k(\mathbb{C}^n)^* \cong \mathbb{C}[z_1, \ldots, z_n]$, i.e., one can say that any element of $\mathrm{S}^k(\mathbb{C}^n)^*$ is a homogeneous polynomial of degree k in n variables. Alternatively, we can define the action of G on $\mathbb{C}[z_1, \ldots, z_n]$ by the map

$$\rho : G \to GL(\mathbb{C}[z_1, \ldots, z_n]) \tag{5.5.2}$$

which is defined as

$$(\rho(g)P)(z) = P(g^{-1}z)$$

where $g \in G$, P is a homogeneous polynomial of degree k and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. For the corresponding Lie algebra $\mathfrak{g}$ of the group G, action is defined as

$$\pi : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}[z_1, \ldots, z_n]);\ (\pi(g)P)(z) = P(-gz). \tag{5.5.3}$$

Consider the dual of the standard representation, i.e., $(\mathbb{C}^n)^*$ of G. All weights of the representation are $-\varepsilon_i$, with highest weight $-\varepsilon_n$ and the highest weight vector is X_n. Now our claim is to show that X_n^k is highest weight vector for $\mathrm{S}^k(\mathbb{C}^n)^*$ with highest weight $-k\varepsilon_n$. Let $Y \in G$ be the Lie group element corresponding to $A \in \mathfrak{g}_+$. Since X_n is highest weight of $(\mathbb{C}^n)^*$ we have $Y \cdot X_n = X_n$. Then

$$\begin{aligned}(Y \cdot X_n^k) &= Y \cdot (X_n \dots X_n) \\ &= (Y \cdot X_n) \dots (Y \cdot X_n) \\ &= (X_n \dots X_n) = X_n^k.\end{aligned}$$

Thus, X_n^k is the highest weight vector as it is fixed by all positive root vectors of G. Let H be the corresponding Lie subgroup of G of Lie subalgebra consisting of diagonal matrices

$$h = \left\{ \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \mid d_i \in \mathbb{C} \right\}. \tag{5.5.4}$$

Now consider the action of H on $V = S^k(\mathbb{C}^n)^*$ to find the highest weight for the highest weight vector X_n^k. Now for $z = c_1e_1 + \cdots + c_ne_n$,

$$(\rho(H)X_n^k)(z) = X_n^k(H^{-1}z). \tag{5.5.5}$$

Here $H = \exp(h) = \operatorname{diag}(e^{d_1}, \dots, e^{d_n})$ then

$$\begin{aligned}(\rho(H)X_n^k)(z) &= X_n^k \begin{pmatrix} e^{-d_1} & 0 & \cdots & 0 \\ 0 & e^{-d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-d_n} \end{pmatrix} \begin{pmatrix} c_1e_1 \\ \vdots \\ c_ne_n \end{pmatrix} \\ &= X_n^k(c_1e^{-d_1}e_1, \dots, c_ne^{-d_n}e_n)^{\mathrm{tr}} \\ &= c_n^k e^{-kd_n} \cdot 1 = \exp(-k\varepsilon_n(h))X_n^k)(z)\end{aligned}$$

where ε_n is defined as in (5.4.2). Thus, $-k\varepsilon_n$ is the highest weight with the highest weight vector X_n^k. Therefore, $\mathrm{S}^k(\mathbb{C}^n)^*$ is an irreducible representation of G as X_n^k is a cyclic vector. □

Let us denote $G(n) = GL(n)$ and $G'(n) = GL(n-1)$.

Theorem 5.5.2. *With the embedding*

$$\begin{aligned} GL(n-1) &\hookrightarrow GL(n) \\ A &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \end{aligned} \tag{5.5.6}$$

$S^k(\mathbb{C}^n)^*$ as a $G'(n)$-module is isomorphic to $\oplus_{i=0}^k S^i(\mathbb{C}^{n-1})^*$.

Proof. To decompose $S^k(\mathbb{C}^n)^*$ as $G'(n)$-module we show the action of $A \in G'(n)$ on any polynomial $P(x_1, \dots, x_n) \in S^i(\mathbb{C}^n)^*$ for $i = 0, \dots, k$ are invariant.

We can write any homogeneous polynomial of degree k in n variables $x_1,\dots,x_n$ as sum of polynomials of the form $x_1^j q_j$, for $j=0,\dots,k$ where q_j is a homogeneous polynomial of degree $k-j$ in variables $x_2,\dots,x_n$. In particular, for $P\in S^k(\mathbb{C}^n)^*$ write

$$P(x_1,\dots,x_n)=\sum_{j=0}^{k} x_1^j q_j,$$

where $q_j\in S^{k-j}((\mathbb{C}^{n-1})^*)$. Now claim is $A\cdot q_j\in S^{k-j}((\mathbb{C}^{n-1})^*)$ for $A\in G'(n)$ with the embedding $\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\in G(n)$. Since $S^k(\mathbb{C}^n)^*$ is an irreducible $G(n)$-module, we have

$$\begin{aligned}\left(\rho\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}P\right)\begin{pmatrix}z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix} &= P\left(\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}^{-1}\begin{pmatrix}z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix}\right)\\ &= x_1^j q_j\begin{pmatrix} z_1\\ A^{-1}\begin{pmatrix}z_2\\ \vdots\\ z_n\end{pmatrix}\end{pmatrix}\in S^k((\mathbb{C}^{n-1})^*). \qquad (5.5.7)\end{aligned}$$

Hence from (5.5.7) x_1^j is evaluated at z_1 and q_j is evaluated at $(z_2',\dots,z_n')$. Thus $q_j\in S^{k-j}((\mathbb{C}^{n-1})^*)$ and $x_1^j\in S^j((\mathbb{C})^*)$ for $j=0,\dots,k$ are invariant under $G'(n)$. Therefore,

$$S^k(\mathbb{C}^n)^*\cong_{G'(n)}\oplus_{i=0}^{k}S^i((\mathbb{C}^{n-1})^*).$$

□

The highest weight of $S^k(\mathbb{C}^n)^*$ is $(0,0,\dots,-k)\in\mathbb{Z}_n$. By Weyl branching law, as $G'(n)$-module the possible highest weights are $(0,0,\dots,-j)\in\mathbb{Z}_{n-1}$ for $j=0,\dots,k$. Hence the corresponding highest weight modules are $S^j((\mathbb{C}^{n-1})^*)$.

Now we have

$$S^k((\mathbb{C}^{n+1})^*)\cong_{G(n)}\oplus_{i=0}^{k}S^i(\mathbb{C}^n)^*.$$

With the embedding (5.5.6)

$$\tau_n: S^k(\mathbb{C}^n)^*\hookrightarrow S^k((\mathbb{C}^{n+1})^*)$$

are $G(n)$-equivariant maps.

Theorem 5.5.3. *With the embedding* (5.3.2)*, suppose* $\mathfrak{g}(\infty)=\varinjlim\mathfrak{g}(n)$*. Let* $S^k(\mathbb{C}^\infty)^*=\varinjlim S^k(\mathbb{C}^n)^*$ *be the direct limit of the direct system* $(\mathbb{N},\tau_n,S^k(\mathbb{C}^n)^*)$.

Then the irreducible $\mathfrak{g}(\infty)$ *module* $S^k(\mathbb{C}^\infty)^*$ *decomposes as* $\mathfrak{g}'(\infty)$ *module* $\oplus_{j=0}^{k}W_j$

where $W_j = \varinjlim W_{n,j}$ *for* $n \geq 2$ *and* j *fixed and*

$$S^k(\mathbb{C}^n)^* = \oplus W_{n,j}$$
$$W_{n,j} := S^j(\mathbb{C})^* \otimes S^{k-j}(\mathbb{C}^{n-1})^*.$$

Proof. For $j = 0, 1, \ldots, k$, a homogeneous polynomial of degree k can be written as sum of polynomials of the form $z_1^j q_j$ where q_j is a homogeneous polynomial of degree $k-j$ in variables $z_2, \ldots, z_n$. Precisely, $S^k(\mathbb{C}^n)^* = \oplus W_{n,j}$ where $W_{n,j} := S^j(\mathbb{C})^* \otimes S^{k-j}(\mathbb{C}^{n-1})^*$. Now each $W_{n,j}$ embeds into $W_{n+1,j}$ by mapping any polynomial $z_1^j q_j$ to itself, i.e.

$$\phi_{n,j} : W_{n,j} \hookrightarrow Wn+1, j$$
$$z_1^j q_j \mapsto z_1^j q_j,$$

where q_j can be interpreted as polynomial in variables $z_2, \ldots, z_n, z_{n+1}$ and the last variable z_{n+1} has degree zero. Then the direct limit $W_j = \varinjlim W_{n,j}$ with respect to the embedding maps $\{\phi_{n,j}\}$ can be identified with the polynomials in infinitely many variables $\{z_1, z_2, \ldots\}$ of the form $z_1^j q_j$ where q_j is homogeneous polynomial of degree $k-j$ in variables $z_2, z_3 \ldots$.

$$S^k(\mathbb{C}^\infty)^* = \oplus_{j=0}^k W_j,$$

by writing a polynomial P of degree k in variables $z_1, z_2, \ldots$ as sum $P = \sum_j z_1^j q_j$ where q_j is of degree $k-j$ in variables $z_2, z_3, \ldots$. Moreover each W_j are irreducible $\mathfrak{g}(\infty - 1)$ representations. □

5.6 Branching law for $\bigwedge^k(\mathbb{C}^\infty)$

Consider $G = GL(n)$ is the Lie group of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$ and $\rho : G \to GL(V)$ is a representation of G for $V = \wedge^k \mathbb{C}^n$.

Theorem 5.6.1. (ρ, V) *is an irreducible G-module.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. Then a basis for the k-th exterior power of $\mathbb{C}^n$ is given by

$$\{e_{i_1} \wedge e_{i_2} \wedge \ldots, \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}. \tag{5.6.1}$$

Now the action of G on V, $\rho : G \to GL(V)$ is defined as

$$\rho(X)(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = Xe_{i_1} \wedge Xe_{i_2} \wedge \cdots \wedge Xe_{i_k}$$

where $X \in G$ and $e_{i_1} \wedge e_{i_2} \wedge \ldots e_{i_k}$ is the basis vectors of $\wedge^k \mathbb{C}^n$. Let X be the Lie group element corresponding to $A \in \mathfrak{g}_+$. Then

$$\rho(X)(e_1 \wedge e_2 \wedge \cdots \wedge e_k) = Xe_1 \wedge Xe_2 \cdots \wedge Xe_k \\ = e_1 \wedge e_2 \cdots \wedge e_k. \tag{5.6.2}$$

Thus, $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is the highest weight vector as it is fixed by all positive root vectors of G. Let H be the Lie subgroup of the Lie group G of the corresponding Lie subalgebra of diagonal matrices $\mathfrak{h}$ given in (5.5.4). Now consider the action of H on $V = \wedge^k \mathbb{C}^n$ to find the highest weight for the highest weight vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$.

$$\begin{aligned}\rho(H)(e_1 \wedge e_2 \wedge \ldots e_k) &= He_1 \wedge \cdots \wedge He_k \\ &= e^{d_1} e_1 \wedge e^{d_2} e_2 \ldots \wedge e^{d_k} e_k \\ &= e^{d_1 + \cdots + d_k}(e_1 \wedge e_2 \cdots \wedge e_k) \\ &= e^{(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k)(h)}(e_1 \wedge e_2 \cdots \wedge e_k)\end{aligned}$$

where ε_i are defined as in (5.4.2). So, $(\varepsilon_1 + \cdots + \varepsilon_k)$ is the highest weight. Finally to show $\wedge^k(\mathbb{C}^n)$ is an irreducible representation of $GL(n,\mathbb{C})$ we have to show that $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a cyclic vector. There always exist a $\mathrm{g} \in G$ such that $\mathrm{g}(e_1) = e_{i_1}, \mathrm{g}(e_2) = e_{i_2}, \cdots, \mathrm{g}(e_k) = e_{i_k}$. So

$$\mathrm{g}(e_1 \wedge e_2 \wedge \cdots \wedge e_k) = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k},$$

which shows $e_1 \wedge e_2 \cdots \wedge e_k$ is cyclic. □

Theorem 5.6.2. *With the embedding map* (5.5.6), $\wedge^k(\mathbb{C}^n)$ *as* $G'(n)$*-module is isomorphic to* $\wedge^{k-1}(\mathbb{C}^{n-1}) \oplus \wedge^k(\mathbb{C}^{n-1})$.

Proof. We have already proved that $\wedge^k \mathbb{C}^n$ is an irreducible G-module. The basis of $\wedge^k(\mathbb{C}^n)$ is given in (5.6.1). To decompose $\wedge^k(\mathbb{C}^n)$ as $G'(n)$-module we have to consider two cases independently, i.e., for $i_1 = 1$ and $i_1 > 1$.

Case 1: $(i_1 = 1)$.

$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ is a basis of $\wedge^k \mathbb{C}^n$ where $2 \leq e_{i_2} < e_{i_3} < \cdots < e_{i_k} \leq n$. Now consider the embedding

$$\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n, \; z \mapsto (0,z).$$

So $\mathrm{span}\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\} \cong_{G'(n)} \wedge^{k-1}(\mathbb{C}^{n-1})$ where the isomorphism is given by

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Here e_1 is $G'(n)$ fixed. For $g \in G(n)$

$$\begin{aligned}g(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}) &= (ge_{i_1} \wedge ge_{i_2} \wedge \cdots \wedge e_{i_k}) \\ &= e_{i_1} \wedge g(e_{i_2} \cdots \wedge e_{i_k}).\end{aligned}$$

Case 2: $(i_1 > 1)$.

$$\text{span}\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\} \cong_{G'(n)} \wedge^k(\mathbb{C}^{n-1}).$$

Hence

$$\wedge^k(\mathbb{C}^n) \cong \wedge^{k-1}(\mathbb{C}^{n-1}) \oplus \wedge^k(\mathbb{C}^{n-1})$$

as $G'(n)$-module. □

The highest weight for $\wedge^k(\mathbb{C}^n)$ is $(1,1,\ldots,1,0,\ldots,0) \in \mathbb{Z}_n$ where first k-th entries are 1. As $G'(n)$-module the possible highest weights are $(1,1,\ldots,1,0,\ldots,0) \in \mathbb{Z}_{n-1}$ where first kth entries are 1 and $(1,1,\ldots,1,0,\ldots,0) \in \mathbb{Z}_{n-1}$ where first $(k-1)$-th entries are 1. The corresponding highest weight modules are $\wedge^k(\mathbb{C}^{n-1})$ and $\wedge^{k-1}(\mathbb{C}^{n-1})$ respectively.

Theorem 5.6.3. *With the embedding* (5.3.2)*, suppose* $\mathfrak{g}(\infty) = \varinjlim \mathfrak{g}(n)$*. Let* $\wedge^k(\mathbb{C}^\infty) = \varinjlim \wedge^k(\mathbb{C}^n)$ *be the direct limit of the direct system* $(\mathbb{N}, \tau_n, \wedge^K\mathbb{C}^n)$*. Then* $\wedge^k(\mathbb{C}^\infty)$ *is a representation of* $\mathfrak{g}(\infty)$*. Further, let* $W^1 := \varinjlim W^1(n)$ *and* $W^2 := \varinjlim W^2(n)$ *where* $W^1(n) = \wedge^{k-1}(\mathbb{C}^{n-1})$ *with morphisms* $p_n : W^1(n) \to W^1(n+1)$ *and* $W^2(n) = \wedge^k(\mathbb{C}^{n-1})$ *with morphisms* $q_n : W^2(n) \to W^2(n+1)$*. Then with the embedding*

$$B \mapsto \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \text{ for } B \in \mathfrak{g}'(n),$$

$\wedge^k(\mathbb{C}^\infty)$ *as* $\mathfrak{g}'(\infty) = \mathfrak{g}(\infty - 1)$*-module, is isomorphic to* $W^1 \oplus W^2$*.*

Proof. Let us denote $V(n) = \wedge^k(\mathbb{C}^n)$ and $V(n+1) = \wedge^k(\mathbb{C}^{n+1})$. Then define

$$\tau_n : V(n) \hookrightarrow V(n+1)$$

such that any element in n variables say $(e_1 \wedge \cdots \wedge e_n) \in V(n)$ adds upto k maps to element in $n+1$ variables with power of e_{n+1} is zero in $V(n+1)$. Consider the map

$$\mathfrak{g}(\infty) \to \mathfrak{gl}(\wedge^k(\mathbb{C}^\infty))$$

and is defined as

$$g(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}) = ge_{i_1} \wedge ge_{i_2} \cdots \wedge ge_{i_k} \tag{5.6.3}$$

Here $g \in \mathfrak{g}(\infty)$ is a matrix with finite number of non-zero entries and $e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}$ where $1 \leq i_1 < i_2 \cdots i_k < \cdots$ are elements in variables $e_1, e_2, \ldots$. The matrix multiplication in (5.6.3) is well defined and $ge_{i_1} \wedge \cdots \wedge ge_{i_k}$ are again elements in variables $e_1, e_2, \ldots$ add upto k. Now for $g_1, g_2 \in \mathfrak{g}(\infty)$

$$\begin{aligned}[\pi_{g_1}, \pi_{g_2}](e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}) &= \pi_{g_1}\pi_{g_2}(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}) - \pi_{g_2}\pi_{g_1}(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}) \\ &= (g_1 g_2 e_{i_1} \wedge \cdots \wedge g_1 g_2 e_{i_k}) - (g_2 g_1 e_{i_1} \wedge \cdots \wedge g_2 g_1 e_{i_k}). \\ &= \pi_{[g_1, g_2]}(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k})\end{aligned}$$

which proves $(\pi, \wedge^k(\mathbb{C}\infty))$ is a representation of $\mathfrak{g}(\infty)$.

Define the $\mathfrak{gl}(n)$-equivariant map $p_n : W^1(n) \to W^1(n+1)$ as

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mapsto e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

assuming $i_1 = 1$ is fixed and $2 \leq i_2 < i_3 \cdots < i_k \leq n-1$ with the power of e_n in $W^1(n+1)$ is zero. Similarly define the map $q_n : W^2(n) \to W^2(n+1)$ as

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

assuming $i_1 \geq 2$ and $2 \leq i_2 < i_3 \cdots < i_k \leq n-1$, with the power of e_n in $W^1(n+1)$ is zero. Now,

$$\begin{aligned}\tau_n(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) &= e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_k} \\ &= p_n(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}) + q_n(e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}).\end{aligned}$$

One can write any elements in variables $z_1, \cdots, z_n$ add upto k as sum of elements $z_1^j q_j$ where $j = 0, 1$. For $j = 1$, q_j is an element in variables $z_2, \ldots, z_n$ add upto $k-1$. For $j = 0, q_j$ is a an element in variables $z_2, \ldots, z_n$ add upto k. So,

$$\wedge^k(\mathbb{C}^n) = W_{n,0} \oplus W_{n,1} \tag{5.6.4}$$

where

$$W_{n,0} = \wedge^0(\mathbb{C}) \otimes \wedge^k(\mathbb{C}^{n-1}) \text{ and } W_{n,1} = \wedge^1(\mathbb{C}) \otimes \wedge^{k-1}(\mathbb{C}^{n-1}). \tag{5.6.5}$$

For $j = 0, 1$ consider the map

$$\begin{aligned}\phi_{n,j} : W_{n,j} &\to W_{n+1,j} \\ z_1^j q_j &\mapsto z_1^j q_j\end{aligned}$$

such that power of z_{n+1} is zero. Therefore for $n \geq 2$ and for $j = 0, 1$

$$W_j = \varinjlim W_{n,j}$$

are the direct limits for the directed systems $(\mathbb{N}, \phi_{n,j}, W_{n,j})$. Now W_0 and W_1 can be identified with elements in variables $\{z_1, z_2, \ldots\}$ add upto k and $k-1$ respectively of the form $z_1^j q_j$ where q_j is an element in variables $\{z_2, z_3, \ldots\}$.

So by writing an element in variables $z_1, z_2, \ldots$ add upto k as $\sum z_1^j q_j$ where q_j are elements in variables $z_2, z_3, \ldots$ add upto $(k-j)$ and denoting $W^1 := W_1$ and $W^2 := W_0$ we have

$$\wedge^k(\mathbb{C}^\infty) \cong W^1 \oplus W^2$$

as $\mathfrak{g}'(\infty)$-module, where each W^1 and W^2 are irreducible $\mathfrak{g}'(\infty)$-modules. □

Now we consider a more general case.

5.7 Branching law for $V_{\wedge_\infty}$

Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h}$ be the Cartan subalgebra with respect to which $\mathfrak{g}$ has a root space decomposition. Here we define a positive root system $\Delta^+ \subset \Delta$ of $\mathfrak{g}$ to be a root system such that $\Delta = \Delta^+ \cup (-\Delta^+)$ and no non-zero sum of positive roots can be equal to a sum of negative roots. Consider a directed family $(\mathfrak{g}_j)_{j\in J}$ of subalgebras of $\mathfrak{g}$ whose union coincides with $\mathfrak{g}(= \varinjlim \mathfrak{g}_j)$. The family is consistent with respect to Δ^+ if the following conditions are satisfied [Nee98]:

1. $\mathfrak{g}_j$ is $\mathfrak{h}$-invariant.
2. $\mathfrak{h}_j := \mathfrak{h} \cap \mathfrak{g}_j$ is maximal abelian in $\mathfrak{g}_j$ has root space decomposition with respect to $\mathfrak{h}_j$.
3. $\Delta^+|_{\mathfrak{h}_j} \cap \Delta_j^+$, where Δ_j^+ is a positive root system of $\mathfrak{g}_j$ with respect to $\mathfrak{h}_j$.

In particular for the directe system of reductive Lie algebras $\mathfrak{g}_n = \mathfrak{gl}(n)$ for $n \in \mathbb{N}$ with the embeddings (5.3.2), let $\mathfrak{g}(\infty) = \varinjlim \mathfrak{g}(n)$ be the direct limit.

Now consider $(V_{\lambda_n})_{n\in\mathbb{N}}$ is a inductive family of highest weight modules of Lie algebras $\mathfrak{g}_n$ with highest weights λ_n with respect to positive system Δ_n^+ and with consistent primitive elements say $(v_{\lambda_n})_{n\in\mathbb{N}}$. So we have the following $\mathfrak{g}(n)$-equivariant embeddings. For each $(n, n+1)$,

$$\begin{aligned}\tau_n : & V_{\lambda_n} \hookrightarrow V_{\lambda_{n+1}} \\ & v_{\lambda_n} \mapsto v_{\lambda_{n+1}}.\end{aligned}$$

Then the linear functionals λ_n are compatible in the sense that they define an element λ_∞ with $\lambda_\infty|_{\mathfrak{h}_n} = \lambda_n$ for all $n \in \mathbb{N}$. Then we consider the inductive limit module V_{λ_∞} of $\mathfrak{g}_\infty$ with highest weight vector $v_{\lambda_\infty} \in \cap_{n\in\mathbb{N}} V_{\lambda_n}$, is the common primitive element for all $\mathfrak{g}$-modules V_{λ_n}. The $\mathfrak{g}_\infty$-module V_{λ_∞} is the irreducible highest weight module with highest weight λ_∞ [Nee98, Nat94].

Consider the highest weight

$$\lambda_\infty = (\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_m, \ldots) \in \mathbb{Z}_\infty,$$

with $\lambda_1 \geq \lambda_2 \geq \cdots, \geq \lambda_m = \lambda_m, \ldots$, whose entries are constant after some $m \in \mathbb{N}$ and say $\lambda_n = (\lambda_1, \ldots, \lambda_n)$. Let $V_{\lambda_{n+1}}$ and V_{λ_n} be irreducible highest weight representations of $\mathfrak{g}(n+1)$ and $\mathfrak{g}'(n+1)$ respectively. Let us denote $\mathfrak{g}'(n) := \mathfrak{gl}(n-1)$ is a subalgebra of $\mathfrak{gl}(n)$ and $\mathfrak{g}'(n+1) := \mathfrak{gl}(n)$. Weyl's branching law tells us which irreducible representations of $\mathfrak{g}'(n)$ embedded into upper left corner of $\mathfrak{g}(n)$ occur in V_{λ_n} and which irreducible representations of $\mathfrak{g}'(n+1)$ embedded into upper left corner of $\mathfrak{g}(n+1)$ occur in $V_{\lambda_{n+1}}$. Suppose we have the following decomposition of $V_{\lambda_{n+1}}$ and V_{λ_n} as $\mathfrak{g}'(n+1)$-module and $\mathfrak{g}'(n)$-module respectively,

$$V_{\lambda_{n+1}} \cong_{\mathfrak{g}'(n+1)} \oplus_{\mu_{n+1}} W_{\mu_{n+1}} \text{ and } V_{\lambda_n} \cong_{\mathfrak{g}'(n)} \oplus_{\mu_n} W_{\mu_n}.$$

Here our claim is τ_n maps W_{μ_n} into $W_{\mu_{n+1}}$. In order to prove this by the $\mathfrak{g}(n)$ eqivariance of τ_n and $\mathfrak{g}(n)$ cyclicity of $\mathfrak{g}'(n)$-module, W_{μ_n} it would be suffice to prove that $\mathfrak{g}'(n)$ highest weight vector of W_{μ_n} maps the $\mathfrak{g}'(n+1)$ highest weight vector of $W_{\mu_{n+1}}$. That is to show that the vectors in V_{λ_n} that are annihilated by positive root vectors in $\mathfrak{g}'(n)$ map by τ_n to vectors in $V_{\lambda_{n+1}}$ that are annihilated by positive root vectors in $\mathfrak{g}'(n+1)$. Also the weight vectors with respect to $\mathfrak{g}'$ are mapped by τ_n to weight vectors with respect to $\mathfrak{g}'(n+1)$.

Let $\mathfrak{g}'$ be the reductive subalgebra of $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}'$, is a subspace of $\mathfrak{h}$ and $\Delta' \subset \Delta$ be the root system for $\mathfrak{g}'$. Again $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ generated by a subset Π' of simple root system Π of $\mathfrak{g}$. Root spaces of $\mathfrak{g}'$ are root spaces of $\mathfrak{g}$. More precisely, consider simple root system $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of $\mathfrak{g}$ and $\mathfrak{g}'$ is the subalgebra generated by subroot system $\Pi' = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ of Π.

Theorem 5.7.1. *If V is highest weight module for $\mathfrak{g} = \mathfrak{gl}(n)$ with highest weight vector v and W is the $\mathfrak{g}' = \mathfrak{gl}(n-1)$ module of V generated by v then $\mathfrak{g}_\beta.W = 0$ for any positive root $\beta \in \Delta^+ \setminus \Delta'$ of $\mathfrak{g}$ which isn't a root of $\mathfrak{g}'$.*

Proof. By the hypothesis, W is the weight space, obtained by applying the negative root vectors of $\mathfrak{g}'$ to v. Hence, our claim is to show

$$E_\beta F_{\alpha_1} F_{\alpha_2} \ldots F_{\alpha_l}(v) = 0,$$

where $\alpha_i \in (\Delta')^+$, $1 \leq i \leq l$. Now we use induction on the length of weight vectors to prove the claim. For $l = 1$,

$$\begin{aligned} E_\beta F_{\alpha_1}(v) &= [E_\beta, F_{\alpha_1}](v) + F_{\alpha_1} E_\beta(v) \\ &= [E_\beta, F_{\alpha_1}](v) \end{aligned}$$

The second term vanishes as v is the highest weight vector of $\mathfrak{g}(n)$, hence annihilated by $\beta \in \Delta \setminus \Delta'$. Again $[E_\beta, F_{\alpha_1}]$ vanishes if $\beta - \alpha_1$ is not a root and we are done. But if $\beta - \alpha_1$

is root then our claim is that it will be a positive root and moreover

$$\beta - \alpha \in \Delta^{+} \setminus \Delta.$$

Since β is a positive root vector can be written as non-negative integral linear combination of simple roots, i.e.

$$\beta = \sum_{i=1}^{m} a_i \alpha_i$$

where $a_i \geq 0$ and atleast one a_i is non-zero. Hence,

$$\begin{aligned} \beta - \alpha_1 &= \sum_{i=1}^{m} a_i \alpha_i - \alpha_1 \\ &= (a_1 - 1)\alpha_1 + \sum_{i=2}^{m} a_i \alpha_i. \end{aligned} \tag{5.7.1}$$

$\beta - \alpha_1$ is either a positive root or a negative root. If it is a negative root then $a_i \leq 0$; $\forall i$ and $a_1 - 1 \leq 0$. So the only possible values for a_i are 0 ; for all $i = 2, \ldots, m$ and for a_1 possible values are 0 and 1. But if $a_1 = 1$ then all the coefficients of $\beta - \alpha_1$ vanishes contradicting the fact that $\beta - \alpha$ is root and $a_1 = 0$ also can't be a case as β is a root. We can conclude $\beta - \alpha_1$ is a positive root. Further β is linear combination of simple roots which are not all simple roots for $\mathfrak{g}'$. Clearly from (5.7.1), $\beta - \alpha_1$ is a linear combination of simple roots of $\mathfrak{g}$. So if at all, $\beta - \alpha$ is a root then $\beta - \alpha \in \Delta^{+} \setminus \Delta'$. So our result is true for $l = 1$. Suppose the result is true for $l = k - 1$.

$$\begin{aligned} E_\beta F_{\alpha_1} \cdots F_{\alpha_{k-1}} F_{\alpha_k}(v) &= [E_\beta, F_{\alpha_1}] F_{\alpha_2} \cdots F_{\alpha_k}(v) + F_{\alpha_1} E_\beta F_{\alpha_2} \cdots F_{\alpha_k}(v) \\ &= [[\cdots[[E_\beta, F_{\alpha_1}]F_{\alpha_2}] \cdots F_{\alpha_{k-1}}]F_k](v) \\ &\quad + F_{\alpha_l}[\ldots[E_\beta, F_{\alpha_1}] \cdots F_{\alpha_{k-1}}](v) \\ &\quad + \cdots + F_{\alpha_1} \cdots F_{\alpha_{k-1}}[E_\beta, F_{\alpha_k}](v) \\ &\quad + F_{\alpha_1} \ldots F_{\alpha_k} E_\beta(v). \end{aligned}$$

By using induction hypothesis all brackets vanish except

$$[[\ldots[[E_\beta, F_{\alpha_1}]F_{\alpha_2}] \ldots F_{\alpha_{k-1}}]F_k] \ldots F_{\alpha_k}], \ldots, [E_\beta, F_{\alpha_k}].$$

But again by the same argument as above these brackets will give positive root vectors those are root vector of $\mathfrak{g}$ but not of $\mathfrak{g}'$ and hence v is annihilated by all those vectors. □

Theorem 5.7.2. *Suppose the irreducible $\mathfrak{g}(n+1)$ representation $V_{\lambda_{n+1}}$ which under $\mathfrak{g}'(n+1)$ decomposes as $\oplus_{\mu_{n+1}} W_{\mu_{n+1}}$. Also irreducible $\mathfrak{g}(n)$ representation V_{λ_n} decomposes under $\mathfrak{g}'(n)$ representation as $\oplus_{\mu_n} W_{\mu_n}$. Then there is a bijection between $\mathfrak{g}'(n)$ subrepresentations of V_{λ_n} and the $\mathfrak{g}'(n+1)$ subrepresentations of $V_{\lambda_{n+1}}$.*

Proof. For a general highest weight $\lambda = (\lambda_1, \lambda_2, \ldots)$ where $\lambda_1 \geq \lambda_2 \geq \ldots$ such that for

$m \in \mathbb{N}$ we have $\lambda_N = \lambda_m$ for $N \geq m$. By using Weyl's branching law for restriction of $\mathfrak{g}$ to $\mathfrak{g}'$, the $\mathfrak{g}'(n)$ highest weights are $(n-1)$ tuples $(\mu_1, \mu_2, \ldots, \mu_{n-1})$ satisfying the condition

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n.$$

So if $\lambda_N = \lambda_m$ for $N \geq m$ we have

$$\lambda_N = \lambda_m \geq \ \mu_N \geq \lambda_{N+1} = \lambda_m.$$

So $\lambda_N = \lambda_m$ when $N \geq m$. Hence there is a bijection between $\mathfrak{g}'(n)$ subrepresentations of V_{λ_n} and the $\mathfrak{g}'(n+1)$ subrepresentations of $V_{\lambda_{n+1}}$ or precisely the bijection between there respective highest weights when $n \geq m+1$ is given by

$$(\mu_1, \mu_2, \ldots, \mu_n, \ldots \mu_m, \ldots, \mu_m) \in \mathbb{Z}_{n-1} \mapsto (\mu_1, \mu_2, \ldots, \mu_m, \ldots, \mu_m, \mu_m) \in \mathbb{Z}_n$$

i.e., the first n tuple on left is sent to the n-tuple on right with adding the last entry as μ_m. This shows the number of irreducible pieces in the decompositions at level n and $n+1$ agree if n is big enough $(n \geq m+1)$ and we naturally identify the corresponding weights occurring at these two levels. □

Now we can iterate this process by replacing $n+1$ and n with $n+2$ and $n+1$ and so on and there by form direct limits of W_{μ_n}.

$$W_\mu := \varinjlim_{n \geq m+1} W_{\mu_n}.$$

Here each W_{μ_n} are $\mathfrak{g}'(n)$-modules. Finally as $\mathfrak{g}'(\infty)$ module

$$V_{\lambda_\infty} \cong \oplus_\mu W_\mu.$$

The above discussion proves the following theorem.

Theorem 5.7.3. *With the embedding* (5.3.2) *let* $\mathfrak{g}(\infty) = \varinjlim \mathfrak{g}(n)$ *be the direct limit. Also* $V_{\lambda_\infty} = \varinjlim V_{\lambda_n}$ *is the direct limit of the direct system* $(\mathbb{N}, \tau_n, V_{\lambda_n})$*, where* τ_n *are the* $\mathfrak{g}(n)$ *equivariant embeddings. Then as* $\mathfrak{g}'(\infty)$*-module* V_{λ_∞} *decomposes as* $\oplus_\mu W_\mu$.

5.8 Conclusion

We have found the branching law for $\mathfrak{g}_\infty$-module V_{λ_∞} with restriction on the highest weight λ_∞ is being bounded. However, one can further study the same when λ_∞ is unbounded, to be precise when all entries of λ_∞ are distinct. Such types of studies are currently in progress. Finally an interesting open problem is to find conditions on pair $(\mathfrak{g}, \mathfrak{g}')$ and on the irreducible highest $\mathfrak{g}$-module V, which guarantee the decomposition of V into countable direct sum of irreducible $\mathfrak{g}'$-module. Here the pairs $(\mathfrak{g}, \mathfrak{g}')$ are infinite dimensional Lie algebras, namely inductive limit of finite dimensional Lie algebras.

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