A Study on Adams Completion and Cocompletion

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A Study on Adams Completion and Cocompletion

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This is to certify that the work presented in the dissertation entitled *A Study on Adams Completion and Cocompletion* submitted by *Snigdha Bharati Choudhury*, Roll Number 512ma6009, is a record of original research carried out by her under my supervision in partial fulfillment of the requirements of the degree of *Doctor of Philosophy* in *Mathematics*. Neither this dissertation nor any part of it has been submitted earlier for any degree or diploma to any institute or university in India or abroad.

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Declaration of Originality

I, *Snigdha Bharati Choudhury*, Roll Number *512ma6009* hereby declare that this dissertation entitled *A Study on Adams Completion and Cocompletion* presents my original work carried out as a doctoral student of NIT Rourkela and, to the best of my knowledge, contains no material previously published or written by another person, nor any material presented by me for the award of any degree or diploma of NIT Rourkela or any other institution. Any contribution made to this research by others, with whom I have worked at NIT Rourkela or elsewhere, is explicitly acknowledged in the dissertation. Works of other authors cited in this dissertation have been duly acknowledged under the sections “References” or “Bibliography”. I have also submitted my original research records to the scrutiny committee for evaluation of my dissertation.

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Abstract

Many algebraic and geometrical constructions from different field of mathematics such as Algebra, Analysis, Topology, Algebraic Topology, Differential Topology, Differentiable Manifolds and so on can be obtained as Adams completion or cocompletion with respect to chosen sets of morphisms in suitable categories. Cayley's Theorem, ascending central series and descending central series are well known facts in the area of group theory. It is shown how these concepts are identified with Adams completion. We obtain a Whitehead-like tower of a module by considering a suitable set of morphisms in the corresponding homotopy category (that is, category of right modules and homotopy module homomorphisms) whose different stages are the Adams cocompletion of the module. Indeed, the work is carried out in a general framework by considering a Serre class of abelian groups. The minimal model of a simply connected differential graded algebra is obtained as the Adams cocompletion with respect to the suitably chosen set of morphisms in the category of 1-connected differential graded algebras over $\mathbb{Q}$ and differential graded algebra homomorphisms. Also with the help of Kopylov and Timofeev result, the relationship between a graph and Adams cocompletion is established.

***Keywords*: *Grothendieck universe*; *Category of fractions*; *Adams completion*; *Adams cocompletion*; *Limit*; *Cayley's theorem*; *Ascending central series*; *Descending central series*; *Homotopy theory of modules*; *Differential graded algebra*; *Minimal model*; *Graph*.**

Contents

Chapter 0

Introduction

Categorical methods of speaking and thinking are turning out to be more widespread in mathematics because they characterize mathematical structure and its ideas in terms of a collection of objects and of arrows (familiar as morphisms). Different authors have depicted the contemplations of complete object and of completion of objects [1] in various categorical or precategorical contexts. In 1973, Adams gave a lucid and compelling analysis of localization and completion and also set up an elegant axiomatic treatment of localization and completion in the framework of category theory and proposed a vast generalization of the existing constructions.

At first the perception of Adams completion, which emerged from a categorical completion process in relation to problems of stability, was introduced by Adams [2–4]. Though the characterization and properties were categorical, the most prominent difficulty in order to deal with it from the categorical viewpoint was due to its topological bounds and set theoretical aspect. At the very beginning, this concept was defined only for some admissible categories and generalized homology or cohomology theories [5–7]. Later on, the same idea was approached broadly by Deleanu, Frei and Hilton [8] because of which it was very convenient to work with an arbitrary category and it's chosen set of morphisms. In addition, they have also suggested the dualization of Adams completion, known to be the Adams cocompletion.

In category theory, the idea of localization [4, 9] is a tool for developing another category from a given one which can be described as follows: a category may have a certain class of morphisms which are not all invertible, despite they ought to be invertible. For instance, one may consider weak homotopy equivalences in the homotopy category of topological spaces: some weak homotopy equivalences are homotopy equivalences and subsequently isomorphisms, yet not every one of them are [10]; on the other hand, two weakly homotopy equivalent spaces behave in completely the same way concerning the properties examined by maps from or to appropriately pleasant spaces and subsequently ought to be ethically isomorphic. So localization of the original category can be framed for a given class of morphisms in a category, which is another category ensuring all ethically invertible morphisms to be invertible, while approximating the original category as nearly as could be expected under the circumstances. A category of fractions is a localization that is developed using a calculus of fractions and its construction is described precisely in [11, 12] which plays a very crucial role in illustrating Adams completion and cocompletion.

Numerous constructions (both algebraic and geometric) from the various fields of mathematics can be demonstrated in terms of Adams completion and cocompletion. The principle part of this thesis is to exhibit some remarkable developments from Algebra, Module Theory, Rational Homotopy Theory and Graph Theory as Adams completion or cocompletion.

Chapter 1 serves as the foundation for the study of the subsequent chapters. It includes some categorical preliminaries like category of fractions, calculus of left (right) fractions, Adams completion (cocompletion). It also includes some results on the existence of Adams completion and cocompletion and their couniversal properties proved by Deleanu, Frei and Hilton, Behera and Nanda etc,.

Cayley's Theorem (named after the British mathematician Arthur Cayley) allows us to know that abstract groups are not distinct from permutation groups. Or maybe, the perspective is distinctive. It basically states that every group is isomorphic to a group of permutation. In Chapter 2, this permutation group is deduced to be the Adams completion of the given group.

In mathematics, basically in the area of Group Theory, the ascending and descending central series (the upper and lower central series respectively) are the most relevant examples of characteristic series which provide a deep understanding to the structure of the group. Chapter 3 is dedicated for relating these two series of a given group with the Adams completion.

In chapter 4, we have recalled the homotopy theory (more specifically the injective homotopy theory) of modules, initially introduced by Peter Hilton [13] and later extensively studied by C. J. Su [14–16]. In [17], Behera and Nanda have obtained the Cartan-Whitehead decomposition of a 0-connected based CW-complex with the help of a suitable set of morphisms whose different stages are precisely the Adams cocompletion; we have used their techniques to study the decomposition of a module. In this chapter, using the injective theory and by considering a Serre class of abelian groups, we have obtained the Cartan-Whitehead-like decomposition of a module.

In 1960, Sullivan proposed the concept of rational homotopy theory; this study depends only on the rational homotopy type of a space or the rational homotopy class of a map. In fact, in rational homotopy theory Sullivan introduced the idea of minimal model [18, 19]. Chapter 5 characterizes the minimal model of a simply connected differential graded algebra in terms of Adams cocompletion with respect to a chosen set of morphisms in the category of 1-connected differential graded algebras over $\mathbb{Q}$ and differential graded algebra homomorphisms.

Recently, graph theory has developed itself as one of the most rapidly growing areas of mathematics. Given any graph G there exists a connected graph H, the center of which is isomorphic to G is an eminent result stated by Kopylov and Timofeev [20]. In Chapter 6, we demonstrate that the center of H is the Adams cocompletion of the given graph G.

Chapter 1

Preliminaries

This chapter is the foundation for the study of the subsequent chapters. It includes the definitions such as category of fractions, calculus of left (right) fractions, Adams completion (cocompletion) etc., and some results on the existence of global Adams completion (cocompletion) of an object in a category $\mathscr{C}$ with respect to a chosen family of morphisms S in $\mathscr{C}$. Also a characterization of Adams completion (cocompletion) in terms of its couniversal property proved by Deleanu, Frei and Hilton is recalled. A stronger version of this result proved by Behera and Nanda [21] is also recalled. Behera and Nanda's result [21] shows that the canonical map from an object to its Adams completion is an element of the set of morphisms under very moderate assumption.

1.1 Category of fractions

In this section we recall the definition of category of fractions and some other definitions relevant to it.

Definition 1.1.1. [12] A *Grothendieck universe* (or simply *universe*) is a collection $\mathscr{U}$ of sets such that the following axioms are satisfied:

U(1): $A \in \mathscr{U} \Longrightarrow A \subset \mathscr{U}$.

U(2): $A \in \mathscr{U}$ and $B \in \mathscr{U} \Longrightarrow \{A, B\} \in \mathscr{U}$.

U(3): $A \in \mathscr{U} \Longrightarrow P(A) \in \mathscr{U}$ (the power set of A is an element of $\mathscr{U}$).

U(4): If $J \in \mathscr{U}$ and if $f : J \to \mathscr{U}$ is a map, then $\bigcup\limits_{j \in J} f(j) \in \mathscr{U}$.

From these conditions one can reach at the following conclusions:

- If $A \in \mathscr{U}$, then every subset of A is also an element of $\mathscr{U}$.
- For any two sets A and B which are elements of $\mathscr{U}$, the sets $A \times B$ and B^A (the set of all maps of A into B) are also in $\mathscr{U}$.
- If J and A_j for each $j \in J$ are elements of $\mathscr{U}$, the product $\prod\limits_{j \in J} A_j$ is an element of $\mathscr{U}$.

The above discussion merges into a solitary sentence, that is, each of the constructions of set theory is carried out with elements of $\mathscr{U}$.

We require the fact that each set is a component of a universe. So for the rest of our study we fix a universe $\mathscr{U}$ containing the set of natural numbers $\mathbb{N}$ (and hence $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$). In the sequel, if we ever work with any universe other than $\mathscr{U}$, then we will indicate explicitly.

Definition 1.1.2. [12] A category $\mathscr{C}$ is said to be *small* (more precisely, *small-$\mathscr{U}$ category*), if the following conditions hold:

S(1): The objects of $\mathscr{C}$ form a set which is an element of $\mathscr{U}$.

S(2): For every pair (X, Y) of objects of $\mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is also an element of $\mathscr{U}$.

Definition 1.1.3. [12] Let $\mathscr{C}$ be any arbitrary category and S a set of morphisms of $\mathscr{C}$. A *category of fractions* of $\mathscr{C}$ with respect to S is a category denoted by $\mathscr{C}[S^{-1}]$ together with a functor

$$F_S : \mathscr{C} \to \mathscr{C}[S^{-1}]$$

having the following properties:

CF(1): For each $s \in S$, $F_S(s)$ is an isomorphism in $\mathscr{C}[S^{-1}]$.

CF(2): F_S is universal with respect to this property : if $G : \mathscr{C} \to \mathscr{D}$ is a functor such that $G(s)$ is an isomorphism in $\mathscr{D}$, for each $s \in S$, then there exists a unique functor $H : \mathscr{C}[S^{-1}] \to \mathscr{D}$ such that $G = HF_S$. Thus we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F_S"] \arrow[d, "G"'] & \mathscr{C}[S^{-1}] \arrow[dl, dashed, "H"] \\ \mathscr{D} & \end{tikzcd}$$

The construction of category of fractions has been described explicitly in [12]. Also it has been observed that both the category $\mathscr{C}[S^{-1}]$ and $\mathscr{C}$ have same objects. Using the notion of calculus of left (right) fractions, category of fractions has been characterized in a very nice way [11, 12].

1.2 Calculus of left (right) fractions

The concept of calculus of left and right fractions have great importance in constructing category of fractions. We recall the definitions and some related results.

Definition 1.2.1. [12] A family of morphisms S in the category $\mathscr{C}$ is said to admit a *calculus of left fractions* if

(a) S is closed under finite compositions and contains identities of $\mathscr{C}$,

(b) any diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \\ Z & \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be completed to a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "t"] \\ Z \arrow[r, "g"'] & W \end{tikzcd}$$

with $t \in S$ and $tf = gs$,

(c) given

$$\begin{tikzcd} X \arrow[r, "s"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \arrow[r, dashed, "t"] & W \end{tikzcd}$$

with $s \in S$ and $fs = gs$, there is a morphism $t : Z \to W$ in S such that $tf = tg$.

The following theorem yields very useful criteria for S to admit a calculus of left fractions.

Theorem 1.2.2. ([8], Theorem 1.3, p.67) *Let S be a closed family of morphisms of $\mathscr{C}$ satisfying*

(a) *if $uv \in S$ and $v \in S$, then $u \in S$,*

(b) *every diagram*

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "s"'] & \bullet \\ \bullet & \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be embedded in a weak push-out diagram

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "s"'] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

with $t \in S$.
Then S admits a calculus of left fractions.

The concept of calculus of right fractions is obtained simply by the dualization of calculus of left fractions.

Definition 1.2.3. [12] A family of morphisms S in a category $\mathscr{C}$ is said to admit a *calculus of right fractions* if

(a) S is closed under finite compositions and contains identities of $\mathscr{C}$,

(b) any diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be completed to a diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd}$$

with $t \in S$ and $ft = sg$,

(c) given

$$\begin{tikzcd} W \arrow[r, dashed, "t"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "s"] & Z \end{tikzcd}$$

with $s \in S$ and $sf = sg$, there is a morphism $t : W \to X$ in S such that $ft = gt$.

In the context of family of morphisms S admitting a calculus of right fractions, the analog of Theorem 1.2.2 imitates instantly by duality.

Theorem 1.2.4. ([8], Theorem 1.3^*, p.70) *Let S be a closed family of morphisms of $\mathscr{C}$ satisfying*

(a) *if $vu \in S$ and $v \in S$, then $u \in S$,*

(b) *any diagram*

$$\begin{tikzcd} & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be embedded in a weak pull-back diagram

$$\begin{tikzcd} \bullet \arrow[r, "t"] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

with $t \in S$.

Then S admits a calculus of right fractions.

The following result will be required in sequel.

Theorem 1.2.5. ([22], Proposition, p.425) *Let $\mathscr{C}$ be a small $\mathscr{U}$-category and S a set of morphisms of $\mathscr{C}$ that admits a calculus of left (right) fractions. Then $\mathscr{C}[S^{-1}]$ is a small $\mathscr{U}$-category.*

1.3 Adams completion and cocompletion

In this section we do reminiscence the abstract definitions of Adams completion and cocompletion.

Definition 1.3.1. [8] Let $\mathscr{C}$ be an arbitrary category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and

$$F : \mathscr{C} \to \mathscr{C}[S^{-1}]$$

be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$,

$$\mathscr{C}[S^{-1}](-, Y) : \mathscr{C} \to \mathscr{S}$$

defines a contravariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, i.e.,

$$\mathscr{C}[S^{-1}](-, Y) \cong \mathscr{C}(-, Y_S),$$

then Y_S is called the (*generalized*) *Adams completion* of Y with respect to the set of morphisms S or simply the S-*completion* of Y. We shall often refer to Y_S as the *completion* of Y.

The idea of Adams cocompletion can be simply obtained by the dualization.

Definition 1.3.2. [8] Let $\mathscr{C}$ be an arbitrary category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and

$$F : \mathscr{C} \to \mathscr{C}[S^{-1}]$$

be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$,

$$\mathscr{C}[S^{-1}](Y, -) : \mathscr{C} \to \mathscr{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, i.e.,

$$\mathscr{C}[S^{-1}](Y, -) \cong \mathscr{C}(Y_S, -),$$

then Y_S is called the (*generalized*) *Adams cocompletion* of Y with respect to the set of morphisms S or simply the S-*cocompletion* of Y. We shall often refer to Y_S as the *cocompletion* of Y.

1.4 Existence theorems

We portray a few results on the presence of Adams completion and cocompletion. We express Deleanu's theorem [23] that under specific conditions, global Adams completion of an object persistently exists.

Theorem 1.4.1. ([23], Theorem 1; [22], Theorem 1) *Let $\mathscr{C}$ be a cocomplete small $\mathscr{U}$-category and S a set of morphisms of $\mathscr{C}$ that admits a calculus of left fractions. Suppose that the following compatibility condition with coproduct is satisfied.*

(C) *If each $s_i : X_i \to Y_i, i \in I$ is an element of S, where the index set I is an element of $\mathscr{U}$, then*

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i \to \bigvee_{i \in I} Y_i$$

is an element of S.

Then every object X of $\mathscr{C}$ has an Adams completion X_S with respect to the set of morphisms S.

Reamrk 1.4.2. Deleanu's theorem cited above has an additional condition to guarantee that $\mathscr{C}[S^{-1}]$ is again a small $\mathscr{U}$-category; in perspective of Theorem 1.2.5 the additional condition is compensated.

The following theorem is an immediate consequence of the dualization of Theorem 1.4.1.

Theorem 1.4.3. ([22], Theorem 2) *Let $\mathscr{C}$ be a complete small $\mathscr{U}$-category and S a set of morphisms of $\mathscr{C}$ that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied.*

(P) *If each $s_i : X_i \to Y_i, i \in I$ is an element of S, where the index set I is an element of $\mathscr{U}$, then*

$$\bigwedge_{i \in I} s_i : \bigwedge_{i \in I} X_i \to \bigwedge_{i \in I} Y_i$$

is an element of S.

Then every object X of $\mathscr{C}$ has an Adams cocompletion X_S with respect to the set of morphisms S.

1.5 Couniversal property

The ideas of Adams completion and cocompletion can be described with the help of a couniversal property which was developed by Deleanu, Frei and Hilton.

Definition 1.5.1. [8] Given a set S of morphisms of $\mathscr{C}$, we define $\bar{S}$, the *saturation* of S, as the set of all morphisms u in $\mathscr{C}$ such that $F_S(u)$ is an isomorphism in $\mathscr{C}[S^{-1}]$. S is said to be *saturated* if $S = \bar{S}$.

The following theorem is evident.

Theorem 1.5.2. ([8], Proposition 1.1, p. 63) *A family S of morphisms of $\mathscr{C}$ is saturated if and only if there exists a functor $F : \mathscr{C} \to \mathscr{D}$ such that S is the collection of all morphisms f such that $F(f)$ is invertible.*

Deleanu, Frei and Hilton have demonstrated that if the set of morphisms S is saturated then the Adams completion of a space is described by a specific couniversal property.

Theorem 1.5.3. ([8], Theorem 1.2, p. 63) *Let S be a saturated family of morphisms of $\mathscr{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathscr{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in S which is couniversal with respect to morphisms of S : given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in S such that $ts = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 1.5.3 can be dualized in the following way.

Theorem 1.5.4. ([8], Theorem 1.4, p. 68) *Let S be a saturated family of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then an object Y_S of $\mathscr{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in S which is couniversal with respect to morphisms of S : given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in S such that $st = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} Y_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & Y \\ Z \arrow[ur, "s"'] & \end{tikzcd}$$

In many case of interests, the set of morphisms S is not saturated. The result stated below, is a more grounded adaptation of Deleanu, Frei and Hilton's characterization of Adams completion in terms of a couniversal property.

Theorem 1.5.5. ([21], Theorem 1.2, p.528) *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathscr{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in $\bar{S}$ which is couniversal with respect to morphisms of S : given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in $\bar{S}$ such that $ts = e$. In other words*, *the following diagram is commutative*:

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 1.5.5 can be dualized in the following way.

Theorem 1.5.6. ([17], Proposition 1.1, p.224) *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then an object Y_S of $\mathscr{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in $\bar{S}$ which is couniversal with respect to morphisms of S : given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in $\bar{S}$ such that $st = e$. In other words*, *the following diagram is commutative*:

$$\begin{tikzcd} Y_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & Y \\ Z \arrow[ur, "s"'] & \end{tikzcd}$$

In the greater interest of the utility it is indispensable for the morphism $e : Y \to Y_S$ ($e : Y_S \to Y$) to be in S; this is the circumstance when S is saturated and the outcome is as stated below.

Theorem 1.5.7. ([8], Theorem 2.9, p.76) *Let S be a saturated family of morphisms of $\mathscr{C}$ and let every object of $\mathscr{C}$ admit an S-completion. Then the morphism $e : Y \to Y_S$ belongs to S and is universal for morphisms to S-complete objects and couniversal for the morphisms in S.*

Dual of the above result states as follows.

Theorem 1.5.8. ([8], dual of Theorem 2.9, p.76) *Let S be a saturated family of morphisms of $\mathscr{C}$ and let every object of $\mathscr{C}$ admit an S-cocompletion. Then the morphism $e : Y_S \to Y$*

belongs to S and is universal for morphisms to S-cocomplete objects and couniversal for the morphisms in S.

In some cases of interests S is not saturated. Under certain assumptions Behera and Nanda have proved an interesting result to show that the morphism $e : Y \to Y_S$ ($e : Y_S \to Y$) always belongs to S, in case S is not saturated.

Theorem 1.5.9. ([21], Theorem 1.3, p.533) *Let S be a set of morphisms in a category $\mathscr{C}$ admitting a calculus of left fractions. Let $e : Y \to Y_S$ be the canonical morphism as defined in Theorem 1.5.5 where Y_S is the S-completion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathscr{C}$ which have the following properties*:

(a) *S_1 and S_2 are closed under composition*;

(b) *$fg \in S_1$ implies that $g \in S_1$*;

(c) *$fg \in S_2$ implies that $f \in S_2$*;

(d) $S = S_1 \cap S_2$.

Then $e \in S$.

The dual of Theorem 1.5.9 states as follows.

Theorem 1.5.10. ([21], dual of Theorem 1.3, p.533) *Let S be a set of morphisms in a category $\mathscr{C}$ admitting a calculus of right fractions. Let $e : Y_S \to Y$ be the canonical morphism as defined in Theorem 1.5.6 where Y_S is the S-cocompletion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathscr{C}$ which have the following properties*:

(a) *S_1 and S_2 are closed under composition*;

(b) *$fg \in S_1$ implies that $g \in S_1$*;

(c) *$fg \in S_2$ implies that $f \in S_2$*;

(d) $S = S_1 \cap S_2$.

Then $e \in S$.

1.6 Limit and Colimit

In this section we recall the universal constructions such as limit and colimit [12, 24, 25].

Definition 1.6.1. [12, 24] Let $\mathscr{C}$ be any $\mathscr{U}$-category and $\mathscr{I}$ be a small indexing $\mathscr{U}$-category. Let $F : \mathscr{I} \to \mathscr{C}$ be a functor. Then $(L, t_i)_{i \in \mathscr{I}}$ is called a *limit* of F if and only if the following conditions hold:

(1) $L \in \mathscr{C}$,

(2) for each $i \in \mathscr{I}$, $t_i : L \to F(i)$ is a morphism in $\mathscr{C}$,

(3) for each morphism $a : i \to j$ in $\mathscr{I}$, the diagram

$$\begin{tikzcd} & F(i) \arrow[dd, "F(a)"] \\ L \arrow[ur, "t_i"] \arrow[dr, "t_j"'] & \\ & F(j) \end{tikzcd}$$

commutes, that is, $F(a)t_i = t_j$,

(4) for any other pair $(X, s_i)_{i\in\mathscr{I}}$ satisfying (1), (2), (3), there exists a unique morphism $\theta : X \to L$ making the following diagram

$$\begin{tikzcd} X \arrow[d, dashed, "\theta"'] \arrow[dr, "s_i"] & \\ L \arrow[r, "t_i"'] & F(i) \end{tikzcd}$$

commutative, that is, $t_i\theta = s_i$ for each $i \in \mathscr{I}$.

The dual concept of limit is colimit.

Definition 1.6.2. [12, 24] Let $\mathscr{C}$ be any $\mathscr{U}$-category and $\mathscr{I}$ be a small indexing $\mathscr{U}$-category. Let $F : \mathscr{I} \to \mathscr{C}$ be a functor. Then $(C, s_i)_{i\in\mathscr{I}}$ is called a *colimit* of F if and only if the following conditions hold:

(1) $C \in \mathscr{C}$,

(2) for each $i \in \mathscr{I}$, $s_i : F(i) \to C$ is a morphism in $\mathscr{C}$,

(3) for each morphism $a : i \to j$ in $\mathscr{I}$, the diagram

$$\begin{tikzcd} F(i) \arrow[dd, "F(a)"'] \arrow[dr, "s_i"] & \\ & C \\ F(j) \arrow[ur, "s_j"'] & \end{tikzcd}$$

commutes, that is, $s_jF(a) = s_i$,

(4) for any other pair $(X, t_i)_{i\in\mathscr{I}}$ satisfying (1), (2), (3), there exists a unique morphism $\theta : C \to X$ making the following diagram

$$\begin{tikzcd} F(i) \arrow[r, "s_i"] \arrow[dr, "t_i"'] & C \arrow[d, dashed, "\theta"] \\ & X \end{tikzcd}$$

commutative, that is $\theta s_i = t_i$ for each $i \in \mathscr{I}$.

1.7 Serre class of abelian groups

The concept of 'getting rid' of troublesome factors in the study of abelian groups is a well known fact. Some of the familiar examples are: by tensoring over $\mathbb{Q}$ or $\mathbb{R}$ to get rid of torsion or by tensoring with $\mathbb{Z}_p$ to get rid of torsions coprime to p and so on. This problem was overcome by Serre, eventually known as Serre class of abelian groups.

Definition 1.7.1. [26] A nonempty class $\mathcal{C}$ of abelian groups is called *Serre class of abelian groups* if whenever the three-term sequence

$$A \to B \to C$$

of abelian groups is exact and $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

An immediate consequence of the above is given as follows.

Theorem 1.7.2. [26] *A class of abelian groups $\mathcal{C}$ is a Serre class iff the following properties are satisfied*:

(a) *$\mathcal{C}$ contains a trivial group.*
(b) *If $A \in \mathcal{C}$ and $A \approx A'$, then $A' \in \mathcal{C}$.*
(c) *If $A \subset B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$ and $B/A \in \mathcal{C}$.*
(d) *If $0 \to A \to B \to C \to 0$ is a short exact sequence with $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.*

Some of the broadly used examples of Serre classes are listed below.

Example 1.7.3. [26]

1. The class of trivial groups.
2. The class of all abelian groups.
3. The class of finite abelian groups.
4. The class of torsion abelian groups.
5. The class of all finitely generated abelian groups.
6. The class of p-groups where p is a prime number.
7. The class of all torsion abelian groups containing no element of order equal to a power of p for a given prime p.

Definition 1.7.4. [26] Let $A, B \in \mathcal{C}$. A homomorphism $f : A \to B$ is a

(a) *$\mathcal{C}$-monomorphism* if ker $f \in \mathcal{C}$.
(b) *$\mathcal{C}$-epimorphism* if coker $f \in \mathcal{C}$.
(c) *$\mathcal{C}$-isomorphism* if it is both $\mathcal{C}$-monomorphism and $\mathcal{C}$-epimorphism.

Definition 1.7.5. [26] Two abelian groups A and B are called *$\mathcal{C}$-isomorphic* if there exists an abelian group C and two $\mathcal{C}$-isomorphisms $f : C \to A$ and $g : C \to B$.

Note 1.7.6. The relation of being $\mathcal{C}$-isomorphic is an equivalence relation.

Theorem 1.7.7. [26, 27] *Let $f : A \to B$ and $g : B \to C$ be homomorphisms of abelian groups. Then the following are always true.*

(a) *If gf is $\mathcal{C}$-monic, then so is f.*

(b) *If gf is $\mathcal{C}$-epic, then so is g.*

(c) *If any two of the three maps f, g and gf are $\mathcal{C}$-isomorphisms, then so is the third.*

The Five lemma is an essential and widely used lemma about commutative diagrams.

Theorem 1.7.8. [28] *Suppose that*

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

be a row exact commutative diagram of abelian groups and homomorphisms. Then the following hold.

(a) *If α is an epimorphism and β and δ are monomorphisms, then γ is a monomorphism.*

(b) *If ε is a monomorphism and β and δ are epimorphisms, then γ is an epimorphism.*

(c) *If α, β, δ and ε are isomorphisms, then γ is an isomorphism.*

Definition 1.7.9. [26] A three-term sequence of groups and homomorphisms

$$A \xleftarrow{f} B \xrightarrow{g} C$$

is said to be $\mathcal{C}$-*exact* if

$$(\text{im } f \cup \ker g)/\text{im } f \in \mathcal{C}$$

and if

$$(\text{im } f \cup \ker g)/\ker g \in \mathcal{C}.$$

Longer sequences are $\mathcal{C}$-*exact* if every three-term sequence is $\mathcal{C}$-exact.

Theorem 1.7.10. [26] *Given any commutative diagram*

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

with $\mathcal{C}$-exact rows such that α, β, δ and ε are $\mathcal{C}$-isomorphisms, then γ is also a $\mathcal{C}$-isomorphism.

Chapter 2

Cayley's Theorem and Adams Completion

There are many fundamental results in group theory which have historical importance. Fundamental Theorem of Group Homorphism has wide application. Lagrange's theorem has been used in numerous applications.

Given any nonempty set, the set of all bijections from the set to itself (also known as the set of all permutations of the set) forms a group under function composition. The resulting group is said to be the symmetric group. This symmetric groups possess subgroups called Sylow subgroups whose characterizations extravagantly appear in literature. The purpose of this chapter is to obtain a characterization of Cayley's theorem. Historically Cayley's theorem is very vital. Groups can arise from groups of permutations. This idea was given by British mathematician Arthur Cayley. Cayley's theorem states that every group is isomorphic to a subgroup of the symmetric group. Mathematicians have studied several characteristics of the Cayley's theorem. We study a categorical aspect of Cayley's theorem. In this chapter we study that this group of permutations in terms of Adams completion.

2.1 Cayley's theorem

From Cayley's Theorem [29] we conclude the following:

Reamrk 2.1.1. Let G be a group. Construct a set $\bar{G}$ as follows:

$$\bar{G} = \{T_g : G \to G \mid T_g(x) = gx \text{ for all } x \in G,\ g \in G\}.$$

It can be easily verified that $\bar{G}$ is a permutation group. Then according to Cayley's theorem G is isomorphic to $\bar{G}$, that is, there exists an isomorphism $\varphi : G \to \bar{G}$.

We need the following result in our sequel.

Theorem 2.1.2. *Let G, $\bar{G}$ and $\varphi : G \to \bar{G}$ be defined as above. If K is a group and $f : G \to K$ is an isomorphism, then there exists a unique isomorphism $\theta : K \to \bar{G}$ such that the diagram below commutes, i.e., $\theta f = \varphi$.*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "f"'] & \bar{G} \\ K \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

Proof. Define $\theta : K \to \bar{G}$ by the rule

$$\theta(k) = \varphi f^{-1}(k)$$

for all $k \in K$. Clearly, θ is well defined and is also a homomorphism. In order to show θ is injective, we have to show ker $\theta = \{e_K\}$. Let $k \in$ ker θ, i.e., $\theta(k) = \varphi f^{-1}(k) = e_{\bar{G}}$. So $f^{-1}(k) = e_G$, i.e., $k = e_K$, showing θ is injective. Next

$$\theta(K) = \varphi f^{-1}(K) = \varphi(G) = \bar{G};$$

so θ is surjective. Thus, θ is an isomorphism. For any $g \in G$,

$$\theta f(g) = \varphi f^{-1}(f(g)) = \varphi(g).$$

Thus $\theta f = \varphi$, i.e., the diagram is commutative. Next we show that θ is unique. Let there exist another $\theta' : K \to \bar{G}$ with $\theta' f = \varphi$. Then for any $k \in K$,

$$\theta(k) = \varphi f^{-1}(k) = \theta' f f^{-1}(k) = \theta'(k).$$

Hence $\theta = \theta'$. □

2.2 The category $\mathscr{G}$

Let $\mathscr{G}$ denote the category of groups and homomorphisms in which the underlying sets of the elements of $\mathscr{G}$ are elements of a fixed Grothendieck universe $\mathscr{U}$. Let us consider a set S which consists of all morphisms $s : P \to Q$ in $\mathscr{G}$ such that s is an isomorphism.

Proposition 2.2.1. *Let $s_i : P_i \to Q_i$ lie in S for each $i \in I$ where the index set I is an element of $\mathscr{U}$. Then*

$$\bigvee_{i\in I} s_i : \bigvee_{i\in I} P_i \to \bigvee_{i\in I} Q_i$$

lies in S.

Proof. Coproducts in $\mathscr{G}$ are the free products. Define a map $s = \bigvee_{i\in I} s_i : P = \bigvee_{i\in I} P_i \to \bigvee_{i\in I} Q_i = Q$ by the rule

$$s(p_1 \cdots p_k) = \varphi(p_1) \cdots \varphi(p_k)$$

where $\varphi(p_j) = s_i(p_j)$ if $p_j \in P_i$ for $j = 1, \cdots, k$. Clearly, s is well defined and is also a homomorphism.

In order to show s is injective we have to show that ker $s = \{e_P\}$. Let $p = p_1 \cdots p_k \in$ ker s, i.e., $s(p_1 \cdots p_k) = e_Q = 1$; this implies $\varphi(p_1) \cdots \varphi(p_k) = 1$ where

$$\varphi(p_j) = s_i(p_j) = \omega_i'(s_i(p_j))$$

for $p_j \in P_i,\ j = 1, \cdots, k$ and $\omega_i' : Q_i \to Q$ defined by

$$\omega_i'(e_{Q_i}) = 1 \quad \text{and} \quad \omega_i'(b) = b$$

for $b \in Q_i$ is a monomorphism for each $i \in I$. Thus

$$\varphi(p_j) = s_i(p_j) = \omega_i'(s_i(p_j)) = 1 = \omega_i'(e_{Q_i})$$

and it follows that $s_i(p_j) = e_{Q_i}$, that is, $p_j = e_{P_i}$ for $p_j \in P_i$ and $j = 1, \cdots, k$. Next let $p_1 \cdots p_k = \eta(p_1) \cdots \eta(p_k)$ where

$$\eta(p_j) = \omega_i(p_j) = \omega_i(e_{P_i}) = 1$$

for $p_j \in P_i$ and $\omega_i : P_i \to P$, defined by

$$\omega_i(e_{P_i}) = 1 \quad \text{and} \quad \omega_i(a) = a$$

for $a \in P_i$, is a monomorphism for each $i \in I$. So $p_1 \cdots p_k = 1 = e_P$. Hence s is injective.

Next let $q_1 \cdots q_k \in Q$ where $q_j \in Q_i$ for $i \in I$ and $j = 1, \cdots, k$. But $q_j = s_i(p_j)$ where $p_j \in P_i$. So $q_1 \cdots q_k = \varphi(p_1) \cdots \varphi(p_k)$ where $\varphi(p_j) = s_i(p_j)$ for $p_j \in P_i$. Hence $q_1 \cdots q_k = s(p_1 \cdots p_k)$, showing s is surjective. Therefore, $s : P \to Q$ is an isomorphism, that is, $s = \bigvee_{i \in I} s_i$ lies in S. □

We will exhibit that the chosen set of morphisms S of the category $\mathscr{G}$ of groups and homomorphisms admits a calculus of left fractions.

Proposition 2.2.2. *S admits a calculus of left fractions.*

Proof. Since S consists of all isomorphisms in $\mathscr{G}$, clearly S is a closed family of morphisms of the category $\mathscr{G}$. We shall verify conditions (i) and (ii) of Theorem 1.2.2. Let $s : P \to Q$ and $t : Q \to R$ be two morphisms in $\mathscr{G}$. We show if $ts \in S$ and $s \in S$, then $t \in S$. Let $q \in \ker t$, i.e., $t(q) = e_R$. So $t(s(p)) = e_R$, $p \in P$. Since ts is an isomorphism we have $p = e_P$. So $q = s(e_P) = e_Q$, i.e., $\ker t = \{e_Q\}$, i.e., t is injective. Since $ts \in S$ and $s \in S$, we have $ts(P) = R$ and $s(P) = Q$. Then $t(Q) = t(s(P)) = R$. So t is surjective. Thus t is an isomorphism, i.e., $t \in S$. Hence condition (i) of Theorem 1.2.2 holds.

In order to prove condition (ii) of Theorem 1.2.2 consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\mathscr{G}$ with $s \in S$. We assert that the above diagram can be completed to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\mathscr{G}$ with $t \in S$. Let

$$D = (B * C)/N,$$

where N is a normal subgroup of $B * C$ generated by

$$\{f(a)s(a)^{-1} : a \in A\}.$$

Define $t : B \to D$ by the rule

$$t(b) = bN$$

for all $b \in B$ and $g : C \to D$ by the rule

$$g(c) = cN$$

for all $c \in C$. Clearly, the two maps are well defined and homomorphisms. For any $a \in A$,

$$tf(a) = f(a)N = s(a)N = gs(a),$$

implies that $tf = gs$. Hence the diagram is commutative.

Next we show $t \in S$, i.e., t is an isomorphism. Take $b \in \ker t$, i.e., $t(b) = e_D = N$; this implies $bN = N$, i.e., $b \in N$. Hence

$$b = f(a)s(a)^{-1} = f(a)s(a^{-1})$$

for some $a \in A$. Now consider the map $\delta_2 : C \to B * C$, defined by

$$\delta_2(e_C) = 1 \quad \text{and} \quad \delta_2(c) = c$$

for $c \in C$; δ_2 is a monomorphism. Then $b1 = f(a)s(a^{-1})$ gives

$$b\delta_2(e_C) = f(a)\delta_2(s(a^{-1})).$$

Hence

$$b = f(a), \quad \delta_2(e_C) = \delta_2(s(a^{-1})).$$

As $\delta_2(e_C) = \delta_2(s(a^{-1}))$, we have $s(a^{-1}) = e_C$, giving $a = e_A$. Then $b = f(e_A) = e_B$, implies that $\ker t = \{e_B\}$, i.e., t is injective.

In order to show t is surjective, take an element $wN \in D$, where $w \in B * C$, and for $w \neq 1$, w can be uniquely written as $w = w_1 \cdots w_k$ where all factors are $\neq 1$ and two adjacent factors do not belong to the same group. Then

$$wN = w_1 \cdots w_k N = w_1 N \cdots w_k N = \varphi(w_1) \cdots \varphi(w_k)$$

where

$$\varphi(w_i) = t(w_i) \ \text{ if } \ w_i \in B$$

and

$$\varphi(w_i) = g(w_i) \ \text{ if } \ w_i \in C.$$

If $w_i \in C$, then

$$w_i = s(a_i)$$

and

$$g(w_i) = g(s(a_i)) = gs(a_i) = tf(a_i).$$

So $wN = t(\text{an element of } B)$, showing t is surjective. Thus t is an isomorphism, i.e., $t \in S$.

Next let $u : B \to X$ and $v : C \to X$ in category $\mathscr{G}$ be such that $uf = vs$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\theta"] & \\
& & X
\end{tikzcd}$$

Define $\theta : D \to X$ by the rule

$$\theta(wN) = \varphi(w_1) \cdots \varphi(w_k), \quad w = w_1 \cdots w_k$$

where

$$\varphi(w_i) = u(w_i) \ \text{ if } \ w_i \in B$$

and

$$\varphi(w_i) = v(w_i) \ \text{ if } \ w_i \in C.$$

We can easily show that θ is well defined and also a homomorphism. Next we show that the two triangles are commutative. For any $b \in B$,

$$\theta t(b) = \theta(bN) = u(b)$$

and for any $c \in C$,

$$\theta g(c) = \theta(cN) = v(c).$$

So $\theta t = u$ and $\theta g = v$. □

The following results are well known.

Proposition 2.2.3. *The category $\mathscr{G}$ is cocomplete.*

Proposition 2.2.4. *S is saturated.*

The category $\mathscr{G}$ and the set of morphims S of $\mathscr{G}$ fulfill all the conditions of Theorem 1.4.1. So from the Theorem 1.5.3, we have the result stated below:

Theorem 2.2.5. *Every object G of the category $\mathscr{G}$ has an Adams completion G_S with respect to the set of morphisms S. Furthermore, there exists a morphism $e : G \to G_S$ in S which is couniversal with respect to the morphisms in S : given a morphism $s : G \to H$ in S there exists a unique morphism $t : H \to G_S$ in S such that $ts = e$. In other words the following diagram is commutative*:

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[d, "s"'] & G_S \\ H \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

2.3 $\bar{G}$ as Adams completion

We show that $\bar{G}$, a permutation group for a group G, is the Adams completion G_S of the group G.

Theorem 2.3.1. $\bar{G} \cong G_S$.

Proof. Consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "e"'] & \bar{G} \\ G_S \arrow[ur, "\theta"'] & \end{tikzcd}$$

By Theorem 2.1.2, there exists a unique morphism $\theta : G_S \to \bar{G}$ in S such that $\theta e = \varphi$.

Next consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[d, "\varphi"'] & G_S \\ \bar{G} \arrow[ur, "\psi"'] & \end{tikzcd}$$

By Theorem 2.2.5, there exists a unique morphism $\psi : \bar{G} \to G_S$ in S such that $\psi\varphi = e$.

From the following diagram

$$
\begin{tikzcd}
G \arrow[rr, "e"] \arrow[dd, "e"'] & & G_S \\
& \bar{G} \arrow[ur, "\psi"] & \\
G_S \arrow[ur, "\theta"] \arrow[uurr, shift right=2ex, "1_{G_S}"'] & &
\end{tikzcd}
$$

we have $\psi\theta e = \psi\varphi = e$. By the uniqueness condition of the couniversal property of e, we conclude $\psi\theta = 1_{G_S}$.

From the following diagram

$$
\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dd, "\varphi"'] & & \bar{G} \\
& G_S \arrow[ur, "\theta"] & \\
\bar{G} \arrow[ur, "\psi"] \arrow[uurr, shift right=2ex, "1_{\bar{G}}"'] & &
\end{tikzcd}
$$

we have $\theta\psi\varphi = \theta e = \varphi$. By the uniqueness condition of the couniversal property of φ, we conclude $\theta\psi = 1_{\bar{G}}$.

Thus $\bar{G} \cong G_S$. $\square$

Chapter 3

Ascending and Descending Central Series in Terms of Adams Completion

There is some relation between the groups and their subgroups. Therefore, the notion of subgroups of a given group can be adopted to study the concept of a series of that group, which gives deep understanding of the structure of the group. Two such familiar series of a group are ascending and descending central series (also known as upper and lower central series respectively), both of which are characteristic series. Despite the names, both of them are central series if and only if the given group is nilpotent. In this chapter, we recall the definition of ascending and descending central series and see how they are related to Adams completion.

Ascending central series and Adams completion

We begin with recalling the definition of ascending central series of a group and perceive how it can be expressed in terms of Adams completion.

3.1 The ascending central series of a group

Subnormal and normal series play a crucial role while studying structure of the groups. It is a well-known fact that every normal series is always subnormal, but the converse need not be true. However, both the notions coincide in case of abelian groups. For our purpose, we will focus on subnormal series.

The most relevant example of subnormal series is ascending central series which can be constructed using the centers of groups. We know that center of a group G, denoted as $Z(G)$, is a normal subgroup of G defined by

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

We recall the concept of ascending central series of a group.

Definition 3.1.1. [30] For any (finite or infinite) group G define the following subgroups inductively:

$$Z_0(G) = 1, \quad Z_1(G) = Z(G)$$

and $Z_{i+1}(G)$ is the subgroup of G containing $Z_i(G)$ such that

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$$

(i.e., $Z_{i+1}(G)$ is the complete preimage in G of the center of $G/Z_i(G)$ under the natural projection). The chain of subgroups

$$Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots$$

is called the *upper central series* or *ascending central series* of G.

3.2 Limit and ascending central series

We recall the concept of limit in the category of groups and homomorphisms in order to establish a couniversal property that will be used in the sequel.

Note 3.2.1. Let $\mathscr{G}$ be the category of groups and homomorphisms and $\mathscr{I}$ be the indexing category whose objects are $0, 1, 2, \cdots$ and morphisms are $a_i : i \to i+1$ for $i \geqslant 0$. Define a functor $F : \mathscr{I} \to \mathscr{G}$ by the rule

$$F(i) = Z_i(G)$$

and

$$F(i \xrightarrow{a_i} i+1) = Z_i(G) \xhookrightarrow{F(a_i)} Z_{i+1}(G)$$

where $F(a_i)$ is an inclusion map. Let us define L as follows:

$$L = \bigcap \{Z_i(G) : i \in \mathscr{I}\} = \{e_G\} = 1.$$

We can readily demonstrate that L is the limit of F (limit of the terms of the ascending central series of the group G). Clearly, the map from G to L is an epimorphism; let us denote it as β.

With the above notations we prove the following result.

Theorem 3.2.2. *If H is a group and $l : G \to H$ is an epimorphism, then there exists a unique epimorphism $\theta : H \to L$ such that $\theta l = \beta$, i.e., the following diagram is commutative*:

$$\begin{tikzcd}
G \arrow[r, "\beta"] \arrow[d, "l"'] & L \\
H \arrow[ur, dashed, "\theta"'] &
\end{tikzcd}$$

Proof. Define $\theta : H \to L$ by the rule

$$\theta(h) = \beta(x)$$

where $h = l(x)$ for some $x \in G$ (as l is an epimorphism). It is easy to show that θ is well defined and an epimorphism. By the definition of θ, the triangle is commutative.

In order to show θ is unique, consider another $\theta' : H \to L$ such that $\theta' l = \beta$. For any $h \in H$, we have

$$\theta(h) = \beta(x) = \theta' l(x) = \theta'(h).$$

So $\theta = \theta'$, showing the uniqueness of θ. □

3.3 The category of groups and homomorphisms

Let $\mathscr{G}$ denote the category of groups and homomorphisms where every element of $\mathscr{G}$ is an element of $\mathscr{U}$. Let S be the set of all morphisms $s : A \to B$ in $\mathscr{G}$ such that s is an epimorphism. For the category $\mathscr{G}$ along with this set of morphisms S, we exhibit the following result.

Proposition 3.3.1. *Let $s_i : A_i \to B_i$ lie in S for each $i \in I$ where the index set I is an element of $\mathscr{U}$. Then*

$$\underset{i\in I}{\vee} s_i : \underset{i\in I}{\vee} A_i \to \underset{i\in I}{\vee} B_i$$

lies in S.

Proof. Coproducts in $\mathscr{G}$ are the free products (usually denoted as $*$). Take $A = \underset{i\in I}{*} A_i$ and $B = \underset{i\in I}{*} B_i$. Define a map $s = \underset{i\in I}{\vee} s_i : A \to B$ by the rule

$$s(a_1 \cdots a_k) = \varphi(a_1) \cdots \varphi(a_k)$$

where $\varphi(a_j) = s_i(a_j)$ for $a_j \in A_i$, $j = 1, \cdots, k$. Clearly, s is well defined and is also a homomorphism. In order to show s is surjective, let $b_1 \cdots b_k \in B$ where $b_j \in B_i$ for $i \in I, j = 1, \cdots, k$. Let $b_j = s_i(a_j)$ where $a_j \in A_i$ (since s_i is surjective). So

$$\begin{aligned} b_1 \cdots b_k &= \varphi(a_1) \cdots \varphi(a_k) \ \ (\text{where } \varphi(a_j) = s_i(a_j) \text{ for } a_j \in A_i) \\ &= s(a_1 \cdots a_k). \end{aligned}$$

Since $a_j \in A_i$ for $i \in I$, we have $a_1 \cdots a_k \in A$. So $s = \underset{i\in I}{\vee} s_i$ lies in S. □

We establish that the chosen set of morphisms S admits a calculus of left fractions.

Proposition 3.3.2. *S admits a calculus of left fractions.*

Proof. Since S is the set of all morphisms $s : A \to B$ in $\mathscr{G}$ such that s is an epimorphism, we have that S is a closed family of morphisms of the category $\mathscr{G}$. We shall verify conditions (i) and (ii) of Theorem 1.2.2. Let $s : A \to B$ and $t : B \to C$ be two morphisms in $\mathscr{G}$. We show that if $ts \in S$ and $s \in S$, then $t \in S$. Take $a_1, a_2 \in A$. Then $ts(a_1a_2) = t(s(a_1)s(a_2))$, which implies $ts(a_1)ts(a_2) = t(s(a_1)s(a_2))$. So t is a homomorphism. Since $ts \in S$ and $s \in S$, we have $ts(A) = C$ and $s(A) = B$. Then $t(B) = t(s(A)) = C$. So t is surjective. Thus $t \in S$. Hence condition (i) of Theorem 1.2.2 holds.

In order to prove condition (ii) of Theorem 1.2.2 consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\mathscr{G}$ with $s \in S$. We assert that the above diagram can be completed to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\mathscr{G}$ with $t \in S$. Let

$$D = (B * C)/N,$$

where N is a normal subgroup of $B * C$ generated by

$$\{f(a)s(a)^{-1} : a \in A\}.$$

Define $t : B \to D$ by the rule

$$t(b) = bN$$

for $b \in B$ and $g : C \to D$ by the rule

$$g(c) = cN$$

for $c \in C$. Clearly, the two maps are well defined homomorphisms. For any $a \in A$,

$$tf(a) = f(a)N = s(a)N = gs(a),$$

implies that $tf = gs$. Hence the diagram is commutative.

Next we show that $t \in S$, i.e., t is an epimorphism. We take an element $wN \in D$, where $w \in B * C$, and for $w \neq 1$, it can be uniquely written as $w = w_1 \cdots w_k$ where all factors are $\neq 1$ and two adjacent factors do not belong to the same group. Then

$$wN = w_1 \cdots w_k N = w_1 N \cdots w_k N = \varphi(w_1) \cdots \varphi(w_k)$$

where

$$\varphi(w_i) = t(w_i) \text{ if } w_i \in B$$

and

$$\varphi(w_i) = g(w_i) \text{ if } w_i \in C.$$

If $w_i \in C$, then $w_i = s(a_i)$ and

$$g(w_i) = g(s(a_i)) = gs(a_i) = tf(a_i),$$

proving $wN = t(\text{an element of } B)$. So t is surjective. Thus $t \in S$.

Next let $u : B \to X$ and $v : C \to X$ in category $\mathscr{G}$ be such that $uf = vs$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\theta"] & \\
 & & X
\end{tikzcd}$$

Define $\theta : D \to X$ by the rule

$$\theta(wN) = \varphi(w_1) \cdots \varphi(w_k), \quad w = w_1 \cdots w_k$$

where

$$\varphi(w_i) = u(w_i) \text{ if } w_i \in B$$

and

$$\varphi(w_i) = v(w_i) \text{ if } w_i \in C.$$

We can easily show that θ is a well defined homomorphism. Next we show that the two triangles of the above diagram are commutative. For any $b \in B$,

$$\theta t(b) = \theta(bN) = u(b)$$

and for any $c \in C$,

$$\theta g(c) = \theta(cN) = v(c).$$

So $\theta t = u$ and $\theta g = v$. □

For the category $\mathscr{G}$ of groups and homomorphisms the following result is trivial.

Proposition 3.3.3. *The category $\mathscr{G}$ is cocomplete.*

From Theorem 1.4.1 and Theorem 1.5.5 we conclude the following.

Theorem 3.3.4. *Every object G of the category $\mathscr{G}$ has an Adams completion G_S with respect to the set of morphisms S. Furthermore, there exists a morphism $e : G \to G_S$ in $\bar{S}$ which is couniversal with respect to the morphisms in S : given a morphism $s : G \to K$ in S there exists a unique morphism $t : K \to G_S$ in $\bar{S}$ such that $ts = e$. In other words the following diagram is commutative*:

$$\begin{tikzcd}
G \arrow[r, "e"] \arrow[d, "s"'] & G_S \\
K \arrow[ur, dashed, "t"'] &
\end{tikzcd}$$

Theorem 1.5.9 shows that the morphism $e : Y \to Y_S$ (as defined in Theorem 1.5.5) always belongs to S (the case where the set of morphisms S is not saturated). However, we come across some cases where $S_1 \subset S_2$ and under this assumption we have $S = S_1 \cap S_2 = S_2$. From Theorem 1.5.9, the following result follows.

Corollary 3.3.5. *Let S be a set of morphisms in a category $\mathscr{C}$ admitting a calculus of left fractions. Let $e : Y \to Y_S$ be the canonical morphism as defined in Theorem 1.5.5, where Y_S is the S-completion of Y. Furthermore, let S have the following properties*:

(a) *S is closed under composition*;

(b) *$fg \in S$ implies that $f \in S$.*

Then $e \in S$.

Proof. Since $F(e)$ is an isomorphism in $\mathscr{C}[S^{-1}]$, assume that $[h, s]$, with $s \in S$, is the inverse of $F(e) = [e, 1_{Y_S}]$. We therefore have a diagram

$$\begin{tikzcd}
 & & Y \arrow[dr, "u"] & \\
Y \arrow[urr, "1_Y"] \arrow[dr, "e"'] & Y_S \arrow[d, "1_{Y_S}"] \arrow[dr, "h"] & Y \arrow[u, "1_Y"'] \arrow[d, "s"] & K \\
 & Y_S \arrow[dr, "h"'] & X \arrow[d, "1_X"] & \\
 & & X \arrow[uur, "v"'] &
\end{tikzcd}$$

with $u = vs \in S$ and $u = vhe$. Moreover, the couniversal property of e implies that we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "u"'] & Y_S \\ K \arrow[ur, "w"'] & \end{tikzcd}$$

So $e = wu = wvhe$ implying that $wvh = 1_{Y_S} \in S$. Condition (b) implies that $w \in S$. Therefore, $e = wu \in S$. □

We show that the morphism $e : G \to G_S$ as constructed in the Theorem 3.3.4 is in S.

Theorem 3.3.6. *The morphism $e : G \to G_S$ is in S.*

Proof. The set of morphisms S satisfies all the conditions of the above corollary. Therefore, $e \in S$. □

3.4 L as Adams completion

In this section we obtain L, the limit of the ascending central series of a group G, as the Adams completion G_S of the group G.

Theorem 3.4.1. *L is the Adams completion of G, that is, $L \cong G_S$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "e"'] & L \\ G_S \arrow[ur, "\theta"'] & \end{tikzcd}$$

By Theorem 3.2.2, there exists a unique morphism $\theta : G_S \to L$ ($\theta \in S \subset \bar{S}$) such that $\theta e = \beta$.

Next consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[d, "\beta"'] & G_S \\ L \arrow[ur, "\psi"'] & \end{tikzcd}$$

By Theorem 3.3.4, there exists a unique morphism $\psi : L \to G_S$ in $\bar{S}$ such that $\psi\beta = e$.

Consider the following diagram:

$$\begin{tikzcd}
G \arrow[rr, "e"] \arrow[dd, "e"'] & & G_S \\
& L \arrow[ur, "\psi"] & \\
G_S \arrow[ur, "\theta"] \arrow[uurr, "1_{G_S}"'] & &
\end{tikzcd}$$

Thus we have $\psi\theta e = \psi\beta = e$. By the uniqueness condition of the couniversal property of e, we conclude $\psi\theta = 1_{G_S}$.

Next consider the following diagram:

$$\begin{tikzcd}
G \arrow[rr, "\beta"] \arrow[dd, "\beta"'] & & L \\
& G_S \arrow[ur, "\theta"] & \\
L \arrow[ur, "\psi"] \arrow[uurr, "1_L"'] & &
\end{tikzcd}$$

Thus we have $\theta\psi\beta = \theta e = \beta$. By the uniqueness condition of the couniversal property of β, we conclude $\theta\psi = 1_L$.

Thus $L \cong G_S$. □

Descending central series and Adams completion

Next we will overview the definition of descending central series of a group (rather a free group) and the theory of associated graded Lie algebra with the descending central series for a free group. Also we deduce the relation with the Adams completion.

3.5 The descending central series of a free group and the associated graded Lie algebra

For a group G and $x, y \in G$, the *commutator* [30] of x, y is defined as

$$[x, y] = x^{-1}y^{-1}xy$$

and for any two subgroups H and K of the group G, their *commutator* [30] is defined to be the following subgroup:

$$[H, K] = \langle\ [h, k] \mid h \in H,\ k \in K\ \rangle$$

The *lower central series* (also known as *descending central series*) of a group G is the descending series of subgroups.

Definition 3.5.1. [31] Let G be a group and $\Gamma_G(k)$ be the k-th term of the lower central series of G defined by

$$\Gamma_G(1) = G,$$
$$\Gamma_G(k) = [\Gamma_G(k-1), G], k \geq 2.$$

It is a well known fact that there is associated a graded Lie algebra with the lower central series of a group. We recall these below.

Definition 3.5.2. [31, 32] For each $k \geq 1$, set $\mathcal{L}_G(k) = \Gamma_G(k)/\Gamma_G(k+1)$ and

$$\mathcal{L}_G = \bigoplus_{k \geq 1} \mathcal{L}_G(k).$$

Then $\mathcal{L}_G$ has a structure of a graded Lie algebra which can be induced from the commutator bracket on G. $\mathcal{L}_G$ is called the *associated Lie algebra* of a group G.

Let us consider the case where G is a free group, say F_n with basis $x_1, \cdots, x_n$. For simplicity, if $G = F_n$, we write $\Gamma_n(k)$, $\mathcal{L}_n(k)$ and $\mathcal{L}_n$ for $\Gamma_G(k)$, $\mathcal{L}_G(k)$ and $\mathcal{L}_G$ respectively. Let H be the abelianization of F_n, i.e.,

$$H = F_n/[F_n, F_n].$$

Then in general, the associated graded Lie algebra $\mathcal{L}_n$ is isomorphic to the free Lie algebra generated by H [31].

Let $\mathcal{L}(H)$ denote the free Lie algebra generated by H. Then $\mathcal{L}_n$ is isomorphic to $\mathcal{L}(H)$. Let $\varphi : \mathcal{L}_n \to \mathcal{L}(H)$ be the isomorphism. For simplicity, we write $\mathcal{L}_n = \mathcal{L}$. The following result is easy to prove.

Theorem 3.5.3. *Let φ, $\mathcal{L}$ and $\mathcal{L}(H)$ be as defined above. If M is a graded Lie algebra and $\psi : \mathcal{L} \to M$ is an isomorphism, then there exists a unique isomorphism $\theta : M \to \mathcal{L}(H)$ making the triangle commutative, i.e., $\theta\psi = \varphi$.*

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \mathcal{L}(H) \\
M \arrow[ur, dashed, "\theta"'] &
\end{tikzcd}$$

Proof. Define $\theta : M \to \mathcal{L}(H)$ by the rule

$$\theta(m) = \varphi\psi^{-1}(m)$$

for all $m \in M$. Clearly, θ is well defined and is also an isomorphism. For any $x \in \mathcal{L}$,

$$\theta\psi(x) = \varphi\psi^{-1}\psi(x) = \varphi(x).$$

Thus $\theta\psi = \varphi$, i.e., the diagram (above) is commutative. Next we show that θ is unique. Let there exist another $\theta' : M \to \mathcal{L}(H)$ with $\theta'\psi = \varphi$. Then for any $m \in M$,

$$\theta(m) = \varphi\psi^{-1}(m) = \theta'\psi\psi^{-1}(m) = \theta'(m).$$

Hence $\theta = \theta'$. □

3.6 The category $\mathscr{GL}$

Let $\mathscr{GL}$ denote the category of graded Lie algebras and graded maps where every element of $\mathscr{GL}$ is an element of $\mathscr{U}$. Let S denote the set of all maps $s : L \to L'$ in $\mathscr{GL}$ such that s is an isomorphism. For this chosen set of morphisms S, the following results hold.

Proposition 3.6.1. *S is saturated.*

Proof. The proof is evident from Theorem 1.5.2. □

Proposition 3.6.2. *Let $s_i : A_i \to B_i$ lie in S for each $i \in I$ where the index set I is an element of $\mathscr{U}$. Then*

$$\bigvee_{i\in I} s_i : \bigvee_{i\in I} A_i \to \bigvee_{i\in I} B_i$$

lies in S.

Proof. The proof is trivial. □

We exhibit that the set of maps S of the category $\mathscr{GL}$ of graded Lie algebras and graded maps admits a calculus of left fractions.

Proposition 3.6.3. *S admits a calculus of left fractions.*

Proof. S is the set of all graded maps in $\mathscr{GL}$ which are isomorphisms. So S is a closed family of morphisms of the category $\mathscr{GL}$. We shall verify conditions (i) and (ii) of Theorem 1.2.2. Let $s : L \to M$ and $t : M \to N$ be two morphisms in $\mathscr{GL}$. We show if $ts \in S$ and $s \in S$, then $t \in S$. Let $m, m' \in M$ and $t(m) = t(m')$. Then as s is an epimorphism, we have

$$m = s(l), \quad m' = s(l')$$

for $l, l' \in L$ and $ts(l) = ts(l')$ implying $l = l'$ as ts is injective. So $s(l) = s(l')$ implies $m = m'$, i.e., t is injective. Next

$$t(M) = t(s(L)) = N,$$

i.e., t is surjective. So $t \in S$. Hence condition (i) of Theorem 1.2.2 holds.

In order to prove condition (ii) of Theorem 1.2.2 consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\mathscr{G}\mathscr{L}$ with $s \in S$. We assert that the above diagram can be completed to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\mathscr{G}\mathscr{L}$ with $t \in S$. Let $B \sqcup C$ denote the free product of B and C. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[r, "j_B"] & B \sqcup C \arrow[d, "q"] \\ C \arrow[r, "j_C"'] & B \sqcup C \arrow[r, "q"'] & D \end{tikzcd}$$

Let N be the ideal generated by

$$\{j_B f(a) - j_C s(a) : a \in A\}$$

and

$$D = (B \sqcup C)/N.$$

Define $t : B \to D$ by the rule

$$t(b) = q j_B(b)$$

for all $b \in B$ and $g : C \to D$ by the rule

$$g(c) = q j_C(c)$$

for all $c \in C$. Clearly, t and g are well defined and graded maps. In [33], it is shown that D is the push-out of $C \xleftarrow{s} A \xrightarrow{f} B$ in the category $\mathscr{G}\mathscr{L}$.

It is left to show that $t \in S$. Let $t(b) = t(b')$ for $b, b' \in B$. Then $q j_B(b) = q j_B(b')$, that is,

$$j_B(b) + N = j_B(b') + N$$

implying $j_B(b) - j_B(b') \in N$. So

$$j_B(b - b') = j_B f(x) - j_C s(x)$$

for some $x \in A$. Then

$$j_B(b-b') + j_C(0) = j_B f(x) + j_C(-s(x))$$

implies $b-b' = f(x), 0 = s(-x)$ gives $x = 0$ and hence $b-b' = f(0) = 0$. So t is injective. Next take an element $w + N \in D$ where w is an element of $B \sqcup C$ other than identity. So

$$\begin{aligned} w + N = q(w') &= qj_B(b) + qj_C(c) \\ &= t(b) + g(c) = t(b) + gs(a) \\ &= t(b) + tf(a) = t(b + f(a)) \end{aligned}$$

shows t is surjective. Hence t is an isomorphism. □

Also from the above discussions the following result follows.

Proposition 3.6.4. *The category $\mathscr{GL}$ is cocomplete.*

For the category $\mathscr{GL}$ and the set of morphisms S of $\mathscr{GL}$, all the conditions of Theorem 1.4.1 are satisfied. So from the Theorem 1.5.3, we have the following.

Theorem 3.6.5. *Every object L of the category $\mathscr{GL}$ has an Adams completion L_S with respect to the set of morphisms S. Furthermore, there exists a morphism $e : L \to L_S$ in S which is couniversal with respect to the morphisms in S : given a morphism $s : L \to M$ in S there exists a unique morphism $t : M \to L_S$ in S such that $ts = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[d, "s"'] & L_S \\ M \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

3.7 $\mathcal{L}(H)$ as Adams completion

We establish that $\mathcal{L}(H)$, the free Lie algebra generated by the abelianization of a free group is the Adams completion of the associated graded Lie algebra $\mathcal{L}$ of the free group.

Theorem 3.7.1. $\mathcal{L}(H) \cong \mathcal{L}_S$.

Proof. Consider the following diagram:

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\varphi"] \arrow[d, "e"'] & \mathcal{L}(H) \\ \mathcal{L}_S \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

By Theorem 3.5.3, there exists a unique morphism $\theta : \mathcal{L}_S \to \mathcal{L}(H)$ in S such that $\theta e = \varphi$.

Next consider the following diagram:

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "e"] \arrow[d, "\varphi"'] & \mathcal{L}_S \\
\mathcal{L}(H) \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

By Theorem 3.6.5, there exists a unique morphism $\alpha : \mathcal{L}(H) \to \mathcal{L}_S$ in S such that $\alpha\varphi = e$.

Consider the diagram:

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "e"] \arrow[dd, "e"'] & & \mathcal{L}_S \\
& \mathcal{L}(H) \arrow[ur, "\alpha"'] & \\
\mathcal{L}_S \arrow[ur, "\theta"'] \arrow[uurr, "1_{\mathcal{L}_S}"'] & &
\end{tikzcd}$$

We have $\alpha\theta e = \alpha\varphi = e$. By the uniqueness condition of the couniversal property of e, we conclude $\alpha\theta = 1_{\mathcal{L}_S}$.

Next consider the diagram:

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "\varphi"] \arrow[dd, "\varphi"'] & & \mathcal{L}(H) \\
& \mathcal{L}_S \arrow[ur, "\theta"'] & \\
\mathcal{L}(H) \arrow[ur, "\alpha"'] \arrow[uurr, "1_{\mathcal{L}(H)}"'] & &
\end{tikzcd}$$

We have $\theta\alpha\varphi = \theta e = \varphi$. By the uniqueness condition of the couniversal property of φ, we conclude $\theta\alpha = 1_{\mathcal{L}(H)}$.

Thus $\mathcal{L}(H) \cong \mathcal{L}_S$. $\square$

Chapter 4

Homotopy Theory of Modules and Adams Cocompletion

The relative homotopy theory of modules, including the (module) homotopy exact sequence was proposed by Peter Hilton. In fact, he has developed homotopy theory in module theory, parallel to the existing homotopy theory in topology. Later homotopy theory of modules was broadly contemplated by C. J. Su [14–16]. In contrast with homotopy theory in general topology, there are two sorts of homotopy theory in module theory, in particular, the injective theory and the projective theory. They are dual however not isomorphic [16].

In this chapter, using injective theory we have obtained the Cartan-Whitehead-like decomposition of a module. We do this in a general framework by considering a Serre class $\mathcal{C}$ of abelian groups [26]. The narrative of homotopy theory of modules, may be assessed from ([13], Chapter 13). We briefly depict a bit of the thoughts towards notational perspectives.

4.1 Homotopy theory in module theory

All through this chapter we will work with right Λ-modules, where Λ is a Dedekind domain [34]. Let M and N be right Λ-modules and $f : M \to N$ a Λ-homomorphism.

Definition 4.1.1. [13] The map f is *i-nullhomotopic*, denoted $f \simeq_i 0$, if f can be extended to some injective module $\overline{M}$ containing M.

Proposition 4.1.2. [13] *The following statements are equivalent.*

(i) $f \simeq_i 0$.

(ii) *f can be extended to every module containing M.*

(iii) *f can be factored through some injective module.*

Proposition 4.1.3. [13] *Let L, P be right Λ-modules and $g : L \to M$, $h : N \to P$ be Λ-homomorphisms. If $f \simeq_i 0$, then $fg \simeq_i 0$ and $hf \simeq_i 0$.*

Proposition 4.1.4. [13] *Let $f' : M \to N$ a Λ-homomorphism. If $f \simeq_i 0$ and $f' \simeq_i 0$, then $f + f' \simeq_i 0$ and $-f \simeq_i 0$.*

In like manner we obtain a subgroup of nullhomotopic homomorphisms

$$\mathrm{Hom}_0(M, N) \subset \mathrm{Hom}_\Lambda(M, N).$$

Definition 4.1.5. [13] The *i-homotopy* group of maps of M to N is

$$\overline{\pi}(M, N) = \mathrm{Hom}_\Lambda(M, N)/\mathrm{Hom}_0(M, N).$$

Let $g : M \to N$ is a Λ-homomorphism. Then $f \simeq_i g$ if $f - g \simeq_i 0$ which is clearly an equivalence relation. The map f is an *i-homotopy equivalence* if there exists a map $h : N \to M$ such that

$$fh \simeq_i 1 : N \to N,$$
$$hf \simeq_i 1 : M \to M.$$

Then we denote $f : M \simeq_i N$ or $M \simeq_i N$. The group $\overline{\pi}(M, N)$ depends only on the equivalence classes of M and N.

We recall the concept of suspension that enriches the motivation of homotopy groups in module theory.

Definition 4.1.6. [13] Consider the short exact sequence

$$0 \to M \to \overline{M} \to \overline{M}/M \to 0$$

where $\overline{M}$ is injective. Then *suspension* of M, denoted as SM, is defined as $\overline{M}/M$ which always has the same homotopy type whatever injective container $\overline{M}$ of M may be chosen. Next the suspension of SM, denoted as S^2M, can be defined in a similar manner and continuing this procedure we will have a sequence

$$SM,\ S^2M,\ \cdots,\ S^nM,\ \cdots$$

which enables us to describe the group $\overline{\pi}(SM, N)$ or more generally $\overline{\pi}(S^nM, N)$. This group, written as $\overline{\pi}_n(M, N)$, is called as the *nth i-homotopy group* of M to N which depends only on the homotopy types of S^nM and N and usually defined by means of an injective resolution of M, namely

$$M \to \overline{M} \to \overline{SM} \to \cdots \to \overline{S^nM} \to \cdots$$

with successive cokernels $SM, S^2M, \cdots, S^{n+1}M$.

4.2 The category $\tilde{\mathscr{M}}$

Let $\mathscr{M}$ denote the category of right Λ-modules and Λ-module homomorphisms and let $\tilde{\mathscr{M}}$ be the corresponding i-homotopy category, that is, the objects of $\tilde{\mathscr{M}}$ are all right Λ-modules and the morphisms are i-homotopy classes of Λ-homomorphisms. For any

Λ-homomorphism $f : M \to N$ where M and N are right Λ-modules, we denote the i-homotopy class of f by $[f]_i$. We assume that the underlying sets of elements of $\mathscr{M}$ are elements of $\mathscr{U}$.

We now choose a suitable set of morphisms S_n for the category $\tilde{\mathscr{M}}$. Let A be any right Λ-module. A morphism $\alpha : X \to Y$ in $\tilde{\mathscr{M}}$ is in S_n if and only if $\alpha_* : \overline{\pi}_m(A, X) \to \overline{\pi}_m(A, Y)$ is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$.

We will demonstrate that the set of morphisms S_n of the category $\tilde{\mathscr{M}}$ admits a calculus of right fractions.

Proposition 4.2.1. *S_n admits a calculus of right fractions.*

Proof. Clearly, S_n is a closed family of morphisms of the category $\tilde{\mathscr{M}}$. We shall verify conditions (i) and (ii) of Theorem 1.2.4.

Let $\alpha : X \to Y$ and $\beta : Y \to Z$ be two morphisms in $\tilde{\mathscr{M}}$. We show if $\beta\alpha \in S_n$ and $\beta \in S_n$, then $\alpha \in S_n$. Since $\beta\alpha \in S_n$ and $\beta \in S_n$, $(\beta\alpha)_* = \beta_*\alpha_* : \overline{\pi}_m(A, X) \to \overline{\pi}_m(A, Z)$ and $\beta_* : \overline{\pi}_m(A, Y) \to \overline{\pi}_m(A, Z)$ are $\mathcal{C}$-isomorphisms for $m > n$ and $\mathcal{C}$-monomorphisms for $m = n$. Therefore α_* is a $\mathcal{C}$-monomorphism for $m \geq n$. In order to show α_* to be a $\mathcal{C}$-isomorphism for $m > n$, we need to show α_* is a $\mathcal{C}$-epimorphism for $m > n$. We have

$$\beta_*\alpha_*(\overline{\pi}_m(A, X)) = \overline{\pi}_m(A, Z)$$

for $m > n$, that is,

$$\beta_*(\alpha_*(\overline{\pi}_m(A, X))) = \beta_*(\overline{\pi}_m(A, Y))$$

for $m > n$. From this we conclude that

$$\alpha_*(\overline{\pi}_m(A, X)) = \overline{\pi}_m(A, Y)$$

for $m > n$, that is, α_* is a $\mathcal{C}$-epimorphism for $m > n$. Therefore, α_* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$. Hence condition (i) of Theorem 1.2.4 holds.

In order to prove the condition (ii) of Theorem 1.2.4 consider the diagram

$$\begin{tikzcd} & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

with $\gamma \in S_n$ in $\tilde{\mathscr{M}}$. We assert that the above diagram can be completed to a weak pull-back diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\beta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

in $\tilde{\mathscr{M}}$ with $\delta \in S_n$. Let $\alpha = [f]_i$ and $\gamma = [s]_i$. We replace f and s by fibrations [35], that is,

$$f = f'r \quad \text{and} \quad s = s't$$

where f', s' are fibrations and r, t are i-homotopy equivalences. Let $\overline{r}$ and $\overline{t}$ be i-homotopy inverses of r and t respectively. Let

$$P_f = X \oplus D \quad \text{and} \quad P_s = Y \oplus D$$

where D is the maximal divisible submodule of Z. Let W be the usual pull-back of f' and s'; hence there exist $p : W \to P_f$ and $q : W \to P_s$ such that $f'p = s'q$. Let $\delta = [\overline{r}p]_i$ and $\beta = [\overline{t}q]_i$. Hence

$$\begin{aligned} \alpha\delta &= [f]_i[\overline{r}p]_i = [f\overline{r}p]_i = [f'r\overline{r}p]_i \\ &= [f'p]_i = [s'q]_i = [s't\overline{t}q]_i \\ &= [s\overline{t}q]_i = [s]_i[\overline{t}q]_i = \gamma\beta. \end{aligned}$$

So we have the following commutative diagram in $\tilde{\mathscr{M}}$.

$$\begin{tikzcd} W \arrow[r, "\delta"] \arrow[d, "\beta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

Moreover, let $\varphi : R \to X$ and $\psi : R \to Y$ in $\tilde{\mathscr{M}}$ be such that $\alpha\varphi = \gamma\psi$. Let $\varphi = [u]_i$ and $\psi = [v]_i$. Thus we have $fu \simeq_i sv$. This implies $f'ru \simeq_i s'tv$, that is, $f'ru - s'tv \simeq_i 0$, that is, $f'ru - s'tv$ can be extended to some injective module $\overline{R}$ containing R.

$$\begin{tikzcd} \overline{R} \arrow[dr, "k"] & \\ R \arrow[u, "h"] \arrow[r, "f'ru - s'tv"'] & Z \end{tikzcd}$$

Thus $kh = f'ru - s'tv$. Consider the following diagram

$$\begin{tikzcd} \overline{R} \arrow[r, dashed, "l"] \arrow[dr, "k"] & P_f \arrow[d, "f'"] \\ R \arrow[u, "h"] \arrow[r, "f'ru - s'tv"'] & Z \end{tikzcd}$$

Since f' is a fibration, there exists $l : \overline{R} \to P_f$ such that $f'l = k$. Thus $f'lh = kh$ and

$$f'ru - s'tv = kh = f'lh,$$

that is,

$$f'(ru - lh) = s'(tv).$$

In the following diagram

$$\begin{tikzcd} R \arrow[dr, dashed, "j"] \arrow[drr, bend left, "ru - lh"] \arrow[ddr, bend right, "tv"'] & & \\ & W \arrow[r, "p"] \arrow[d, "q"] & P_f \arrow[d, "f'"] \\ & P_s \arrow[r, "s'"'] & Z \end{tikzcd}$$

since W is the pull-back of f' and s' in $\mathscr{M}$, there exists $j : R \to W$ such that

$$pj = ru - lh \;\text{ and }\; qj = tv.$$

Let $\theta = [j]_i$. In the following diagram in $\tilde{\mathscr{M}}$,

$$\begin{tikzcd} R \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\psi"'] & & \\ & W \arrow[r, "\delta"] \arrow[d, "\beta"] & X \arrow[d, "\alpha"] \\ & Y \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

we have

$$\delta\theta = [\overline{r}p]_i[j]_i = [\overline{r}pj]_i = [\overline{r}(ru - lh)]_i = [\overline{r}ru - \overline{r}lh]_i = [u - \overline{r}lh]_i.$$

We claim that $[u - \overline{r}lh]_i = [u]_i$, that is, $u - \overline{r}lh \simeq_i u$; hence we need to show that $\overline{r}lh \simeq_i 0$, which is evident from the following commutative diagram.

$$\begin{tikzcd} \overline{R} \arrow[dr, "\overline{r}l"] & \\ R \arrow[u, "h"] \arrow[r, "\overline{r}lh"'] & X \end{tikzcd}$$

Also

$$\beta\theta = [\bar{t}q]_i[j]_i = [\bar{t}qj]_i = [\bar{t}tv]_i = [v]_i = \psi.$$

Thus we have the required pull-back diagram in $\tilde{\mathscr{M}}$.

It remains to show that $\delta \in S_n$. Let $F = \ker \beta$ and from the commutative diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ W \arrow[r, "\delta"] \arrow[d, "\beta"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

in $\tilde{\mathscr{M}}$ we have the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \overline{\pi}_{m+1}(A,Y) \arrow[r] \arrow[d, "\gamma_*"'] & \overline{\pi}_m(A,F) \arrow[r] \arrow[d, equal] & \overline{\pi}_m(A,W) \arrow[r] \arrow[d, "\delta_*"] & {} \\ \cdots \arrow[r] & \overline{\pi}_{m+1}(A,Z) \arrow[r] & \overline{\pi}_m(A,F) \arrow[r] & \overline{\pi}_m(A,X) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} \overline{\pi}_m(A,Y) \arrow[r] \arrow[d, "\gamma_*"'] & \overline{\pi}_{m-1}(A,F) \arrow[r] \arrow[d, equal] & \cdots \\ \overline{\pi}_m(A,Z) \arrow[r] & \overline{\pi}_{m-1}(A,F) \arrow[r] & \cdots \end{tikzcd}$$

By Five Lemma, δ_* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$, that is, $\delta \in S_n$. □

The following result holds for the category $\tilde{\mathscr{M}}$ together with the chosen set of morphisms S_n.

Proposition 4.2.2. *Let $s_j : X_j \to Y_j$ lie in S_n for each $j \in J$ where the index set J is an element of $\mathscr{U}$. Then*

$$\bigwedge_{j\in J} s_j : \bigwedge_{j\in J} X_j \to \bigwedge_{j\in J} Y_j$$

lies in S_n.

Proof. Let

$$s = \prod_{j \in J} s_j, \quad X = \prod_{j \in J} X_j \quad \text{and} \quad Y = \prod_{j \in J} Y_j.$$

Define a map $s : X \to Y$ by the rule

$$s(x) = (s_j(x_j))_{j \in J}$$

where $x = (x_j)_{j \in J}$. Clearly, s is well defined and is also a morphism in $\tilde{\mathscr{M}}$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "p_j"'] & Y \arrow[d, "q_j"] \\ X_j \arrow[r, "s_j"'] & Y_j \end{tikzcd}$$

where p_j and q_j are the projections. Let $F = \ker p_j$ and from the commutative diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ X \arrow[r, "s"] \arrow[d, "p_j"'] & Y \arrow[d, "q_j"] \\ X_j \arrow[r, "s_j"'] & Y_j \end{tikzcd}$$

we have the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \overline{\pi}_{m+1}(A, X_j) \arrow[r] \arrow[d, "s_{j*}"'] & \overline{\pi}_m(A, F) \arrow[r] \arrow[d, equal] & \overline{\pi}_m(A, X) \arrow[r] \arrow[d, "s_*"] & {} \\ \cdots \arrow[r] & \overline{\pi}_{m+1}(A, Y_j) \arrow[r] & \overline{\pi}_m(A, F) \arrow[r] & \overline{\pi}_m(A, Y) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} \overline{\pi}_m(A, X_j) \arrow[r] \arrow[d, "s_{j*}"'] & \overline{\pi}_{m-1}(A, F) \arrow[r] \arrow[d, equal] & \cdots \\ \overline{\pi}_m(A, Y_j) \arrow[r] & \overline{\pi}_{m-1}(A, F) \arrow[r] & \cdots \end{tikzcd}$$

By Five Lemma, s_* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$, that is, $s \in S_n$. $\square$

The following result is well known.

Proposition 4.2.3. *The category $\tilde{\mathscr{M}}$ is complete.*

4.3 Existence of Adams cocompletion in $\tilde{\mathscr{M}}$

Using Propositions 4.2.1 - 4.2.3 and Theorem 1.4.3 we reach at the accompanying result.

Theorem 4.3.1. *Every object X in the category $\tilde{\mathscr{M}}$ has an Adams cocompletion X_{S_n} with respect to the set of morphisms S_n.*

Since every object in the category $\tilde{\mathscr{M}}$ has Adams cocompletion with respect to the set of morphisms S_n, from Theorem 1.5.6 we conclude the following result.

Theorem 4.3.2. *Every object X of the category $\tilde{\mathscr{M}}$ has an S_n cocompletion with respect to the set of morphisms S_n if and only if there exists a morphism $e_n : X_{S_n} \to X$ in $\overline{S}_n$ which is couniversal with respect to the morphisms in S_n : given a morphism $s : Y \to X$ in S_n there exists a unique morphism $t_n : X_{S_n} \to Y$ in $\overline{S}_n$ such that $st_n = e_n$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} X_{S_n} \arrow[r, "e_n"] \arrow[d, dashed, "t_n"'] & X \\ Y \arrow[ur, "s"'] & \end{tikzcd}$$

We prove that the morphism $e_n : X_{S_n} \to X$ as constructed above is in S_n.

Theorem 4.3.3. $e_n \in S_n$.

Proof. Let

$S_n^1 = \{\alpha : X \to Y$ in $\tilde{\mathscr{M}} \mid \alpha_* : \overline{\pi}_m(A, X) \to \overline{\pi}_m(A, Y)$ is a $\mathcal{C}$-monomorphism for $m \geq n\}$

and

$S_n^2 = \{\alpha : X \to Y$ in $\tilde{\mathscr{M}} \mid \alpha_* : \overline{\pi}_m(A, X) \to \overline{\pi}_m(A, Y)$ is a $\mathcal{C}$-epimorphism for $m > n\}$.

Clearly,

$$S_n = S_n^1 \cap S_n^2$$

and S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.5.10. Hence $e_n \in S_n$. $\square$

Behera and Nanda [17] have obtained the Cartan-Whitehead decomposition of a 0-connected based CW-complex with the help of a suitable set of morphisms. Following techniques of the works of Behera and Nanda [17] we obtain a Whitehead-like tower for a module with the help of chosen set of morphisms S_n whose different stages are the Adams cocompletion with respect to the set of morphisms S_n.

Theorem 4.3.4. *Let X be a right Λ-module. Then for $n \geq 0$, there exists right Λ-modules X_{S_n}, maps $e_n : X_{S_n} \to X$ and maps $\theta_{n+1} : X_{S_{n+1}} \to X_{S_n}$ such that*

(i) $e_{n*} : \overline{\pi}_m(A, X_{S_n}) \to \overline{\pi}_m(A, X)$ *is a $\mathcal{C}$-isomorphism for $m > n$ and* $\overline{\pi}_m(A, X_{S_n}) = 0$ *for $m \leq n$.*

(ii) $e_{n+1} = e_n \circ \theta_{n+1}$.

Proof. For every $n \geq 0$, let X_{S_n} be the S_n-cocompletion of X and $e_n : X_{S_n} \to X$ be the canonical map. We have already shown $e_n \in S_n$. So

$$e_{n*} : \overline{\pi}_m(A, X_{S_n}) \to \overline{\pi}_m(A, X)$$

is a $\mathcal{C}$-isomorphism for $m > n$. Every module has an injective resolution [34]. Consider an injective resolution of A as

$$A \to \overline{A} \to \overline{SA} \to \cdots \to \overline{S^m A} \to \cdots$$

with successive cokernels

$$SA,\ S^2A,\ \cdots,\ S^{m+1}A,\ \cdots.$$

We claim that $S^m A$ is injective. We can decompose the above sequence into the following short exact sequences.

$$0 \to A \to \overline{A} \to SA \to 0$$

$$0 \to SA \to \overline{SA} \to S^2A \to 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$0 \to S^{m-1}A \to \overline{S^{m-1}A} \to S^m A \to 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

Applying $\mathrm{Ext}^j_\Lambda(A, -)$ for every integer $j \geq 1$ to the short exact sequence

$$0 \to S^{m-1}A \to \overline{S^{m-1}A} \to S^m A \to 0,$$

we get the following short exact sequence [34].

$$0 \to \mathrm{Ext}^j_\Lambda(A, S^{m-1}A) \to \mathrm{Ext}^j_\Lambda(A, \overline{S^{m-1}A}) \to \mathrm{Ext}^j_\Lambda(A, S^m A) \to 0$$

Since $\overline{S^{m-1}A}$ is injective, $\mathrm{Ext}^j_\Lambda(A, \overline{S^{m-1}A}) = 0$ for every integer $j \geq 1$ [34]. So $\mathrm{Ext}^j_\Lambda(A, S^m A) = 0$ for every integer $j \geq 1$. This concludes $S^m A$ is injective [34]. Therefore, $\overline{\pi}_m(A, X_{S_n}) = 0$ for $m \leq n$ [13].

Next we have $e_n \in S_n \subset S_{n+1}$. By the couniversal property of e_{n+1} there exists a unique morphism $\theta_{n+1} : X_{S_{n+1}} \to X_{S_n}$ such that the following diagram

$$\begin{tikzcd}
X_{S_{n+1}} \arrow[r, "e_{n+1}"] \arrow[d, dashed, "\theta_{n+1}"'] & X \\
X_{S_n} \arrow[ur, "e_n"'] &
\end{tikzcd}$$

commutes, that is, $e_{n+1} = e_n \circ \theta_{n+1}$. Thus we get a Whitehead-like tower of a module in the category $\tilde{\mathscr{M}}$.

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X_{S_{n+1}} \arrow[d, "\theta_{n+1}"'] \arrow[ddddr, "e_{n+1}"] & \\
X_{S_n} \arrow[d] \arrow[dddr, "e_n"'] & \\
\vdots \arrow[d] & \\
X_{S_1} \arrow[d, "\theta_1"'] \arrow[dr, "e_1"'] & \\
X_{S_0} \arrow[r, "e_0"'] & X
\end{tikzcd}$$

$\square$

Chapter 5

Minimal Model as Adams Cocompletion

Rational homotopy theory, which is the study of properties that depend only on the rational homotopy type of space or the rational homotopy class of a map, was introduced by Sullivan in 1960. Initially, rational homotopy theory arose from an underlying geometrical construction. Rational homotopy theory is less complicated than ordinary theory and is remarkably computational because of an explicit algebraic formulation revealed by Quillen [36] and Sullivan [18]. In rational homotopy theory Sullivan introduced the idea of minimal model.

In [21], Behera and Nanda have obtained the various stages of the Postnikov decomposition of a 1-connected CW-complex in terms of Adams completion of the space. Minimal model may be treated as the dual version of Postnikov decomposition. Adams cocompletion is the dual concept of Adams completion. It is natural for someone to conclude that minimal model can be expressed in terms of Adams cocompletion. In [37], the minimal model of a simply connected differential graded algebra is characterized in terms of Adams cocompletion under certain assumptions. In this chapter, by dropping the assumptions of [37] we express minimal model in terms of Adams cocompletion. More elaborately, we show that the minimal model of a simply connected differential graded algebra can be expressed as the Adams cocompletion of the given simply connected differential graded algebra in the category of 1-connected differential graded algebras (in short d.g.a.'s) over $\mathbb{Q}$ and d.g.a.-homomorphisms.

5.1 Minimal model

We reminisce a couple of definitions that are key for our outcome.

Definition 5.1.1. [38] Let $\mathbb{Q}$ denote the set of rational numbers. By a *graded algebra*, A, over $\mathbb{Q}$ we mean a graded $\mathbb{Q}$-vector space

$$A = \underset{n\geq 0}{\oplus} A^n$$

together with an associative multiplication

$$\mu : A \otimes A \to A$$

which is graded, that is,

$$\mu(A^n \otimes A^m) \subset A^{n+m}$$

and graded commutative, that is,

$$a \cdot b = (-1)^{nm} b \cdot a \text{ when } a \in A^n \text{ and } b \in A^m.$$

We also assume, unless otherwise stated, that A has an identity element $1 \in A^0$. The elements of A^n are said to be *homogeneous of degree* n (or *dimension* n).

Definition 5.1.2. [38] A *differential graded algebra* A is graded algebra, together with a differential, d, of degree $+1$ which is a derivation. This means that for each n there is a vector space homomorphism

$$d = d_n : A^n \to A^{n+1}$$

satisfying

(i) $d \circ d = 0$ (differential);

(ii) $d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b)$ for $a, b \in A^n$ (derivation).

Definition 5.1.3. [38] If A is a differential graded algebra, let

$$Z^n(A) = \text{Ker } \{d : A^n \to A^{n+1}\} = \text{Subspace of cocycles of } A^n,$$
$$B^n(A) = \text{Im } \{d : A^{n-1} \to A^n\} = \text{Subspace of coboundaries of } A^n.$$

Then $Z^*(A)$ and $B^*(A)$ are defined as

$$Z^*(A) = \bigoplus_{n \geq 0} Z^n(A) \quad \text{and} \quad B^*(A) = \bigoplus_{n \geq 0} B^n(A).$$

The proof of the following result is immediate.

Proposition 5.1.4. $B^n(A) \subset Z^n(A)$.

Proof. Let $x \in B^n(A)$. Then $x = d^{n-1}(y)$ where $y \in A^{n-1}$. Now

$$d^n(x) = d^n d^{n-1}(y) = 0.$$

So $x \in$ Ker d^n, that is, $x \in Z^n(A)$. Thus $B^n(A) \subset Z^n(A)$. □

Definition 5.1.5. [38] The n*th cohomology space* of A, denoted as $H^n(A)$, is defined to be the quotient vector space

$$H^n(A) = Z^n(A)/B^n(A).$$

As d is a derivation, we see that $Z^*(A)$ is a subalgebra of A and $B^*(A)$ is an ideal in $Z^*(A)$. Hence

$$H^*(A) = \bigoplus_{n \geq 0} H^n(A) = Z^*(A)/B^*(A)$$

is a graded algebra, called the *cohomology algebra* of A.

Definition 5.1.6. [38] A differential graded algebra A is is said to be

(i) *connected* if $H^0(A) = \mathbb{Q}$;

(ii) 1-*connected* or *simply connected* if it is connected and $H^1(A) = 0$.

Definition 5.1.7. [38] If A and B are graded algebras, a function $f : A \to B$ is a *graded algebra homomorphism* if it preserves all the algebraic structure, that is,

(i) $f(A^n) \subset B^n$,

(ii) $f(a + b) = f(a) + f(b)$,

(iii) $f(a \cdot b) = f(a) \cdot f(b)$.

We also assume that $f(1) = 1$.

Definition 5.1.8. [38] If A and B are differential graded algebras, then $f : A \to B$ is a *differential graded algebra homomorphism* if

(i) f is a graded algebra homomorphism,

(ii) f commutes with the differentials, i.e., $f \circ d_A = d_B \circ f$.

Definition 5.1.9. [38] If $f : A \to B$ is a differential graded algebra homomorphism then f *induces a map*

$$f^* : H^*(A) \to H^*(B)$$

defined by the rule

$$f^*([z]) = f^*(z + B(A)) = f(z) + B(B) = [f(z)]$$

where $[z]$ denotes the cohomology class of the element $z \in Z^*(A)$. Clearly, f^* is a homomorphism of graded algebras.

Definition 5.1.10. [38] If $A = \underset{n \geq 0}{\oplus} A^n$ is a graded algebra, set

$$A^+ = \underset{n \geq 1}{\oplus} A^n.$$

Define $D(A)$ to be the image of $A^+ \otimes A^+$ under multiplication. $D(A)$ is clearly an ideal of A, called the *ideal of decomposables*; it consists of all sums of non-trivial products in A.

Definition 5.1.11. [38] If A is a differential graded algebra, we say that A has a *decomposable differential* if the image of the differential is contained in the ideal of decomposables, that is,

$$B^*(A) \subset D(A).$$

Definition 5.1.12. [38] A differential graded algebra M is called a *minimal algebra* if it satisfies the following properties:

(i) M is free as a graded algebra;

(ii) M has a decomposable differential;

(iii) $M^0 = \mathbb{Q}$, $M^1 = 0$;

(iv) M has cohomology of finite type, that is, for each n, $H^n(M)$ is a finite dimensional vector space.

Note that Properties (ii)-(iv) imply

(v) for each n, M^n is a finite dimensional vector space.

Let $\mathscr{DGA}$ be the category of differential graded algebras and differential graded algebra homomorphisms and $\mathscr{M}$ denote the full subcategory of $\mathscr{DGA}$ consisting of all minimal algebras and all differential graded algebra maps between them.

Definition 5.1.13. [38] Suppose A is a simply connected differential graded algebra. A differential graded algebra $M = M(A)$ is called a *minimal model* for A if

(i) $M \in \mathscr{M}$;

(ii) there is a differential graded algebra map $\rho : M \to A$ which induces an isomorphism on cohomology

$$\rho^* : H^*(M) \xrightarrow{\cong} H^*(A).$$

The following result will be used in our work.

Theorem 5.1.14. [38, 39] *Let A be a simply connected differential graded algebra over $\mathbb{Q}$ and $M = M(A)$ be a minimal model for A. Then the map $\rho : M \to A$ induces an isomorphism on cohomology in all dimensions. Then ρ has following couniversal property*: *for any simply connected differential graded algebra C over $\mathbb{Q}$ and differential graded algebra map $\eta : C \to A$ which induces an isomorphism on cohomology in all dimensions*, *there exists a differential graded algebra map $\varphi : M \to C$ such that $\eta\varphi \simeq \rho$, that is*, *the following diagram commutes*:

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, dashed, "\varphi"'] & A \\
C \arrow[ur, "\eta"'] &
\end{tikzcd}$$

5.2 The category $\mathscr{D}$

Let $\mathscr{D}$ be the category of 1-connected differential graded algebras (in short d.g.a.'s) over $\mathbb{Q}$ and d.g.a.-homomorphisms where every element of $\mathscr{D}$ is an element of $\mathscr{U}$. Let S be the set of all d.g.a.-homomorphisms which induce cohomology isomorphisms in all dimensions.

The going with result is significant for the set of morphisms S for the category $\mathscr{D}$.

Proposition 5.2.1. *S is saturated.*

Proof. The proof follows from Theorem 1.5.2. □

We will show that the set of morphisms S of the category $\mathscr{D}$ admits a calculus of right fractions.

Proposition 5.2.2. *The set of morphisms S admits a calculus of right fractions.*

Proof. Clearly, S is a closed family of morphisms of the category $\mathscr{D}$. We shall verify conditions (i) and (ii) of Theorem 1.2.4. Let u, v be two morphisms in $\mathscr{D}$. We show that if $vu \in S$ and $v \in S$, then $u \in S$. We know $(vu)^* = v^*u^*$ and v^* are both cohomology isomorphisms implying u^* is a cohomology isomorphism. Thus $u \in S$. Hence condition (i) of Theorem 1.2.4 holds.

In order to prove condition (ii) of Theorem 1.2.4 consider the diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

in $\mathscr{D}$ with $s \in S$. We assert that the above diagram can be completed to a weak pull-back diagram

$$\begin{tikzcd} D \arrow[r, dashed, "t"] \arrow[d, dashed, "g"'] & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

in $\mathscr{D}$ with $t \in S$. Since A, B and C are in $\mathscr{D}$,

$$A = \bigoplus_{n \geq 0} A^n, \quad B = \bigoplus_{n \geq 0} B^n, \quad C = \bigoplus_{n \geq 0} C^n,$$

and

$$f = \bigoplus_{n \geq 0} f^n, \quad s = \bigoplus_{n \geq 0} s^n$$

where

$$f^n : A^n \to B^n \quad \text{and} \quad s^n : C^n \to B^n$$

are d.g.a. homomorphisms. Let

$$D^n = \{(a, c) \in A^n \times C^n : f^n(a) = s^n(c)\}.$$

Let $t^n : D^n \to A^n$ be defined by the rule

$$t^n(a, c) = a$$

and $g^n : D^n \to C^n$ be defined by the rule

$$g^n(a,c) = c.$$

Clearly, t^n and g^n are d.g.a. homomorphisms. Let

$$D = \underset{n\geq 0}{\oplus} D^n, \quad t = \underset{n\geq 0}{\oplus} t^n \quad \text{and} \quad g = \underset{n\geq 0}{\oplus} g^n.$$

Clearly, the above diagram is commutative. Here we have to show D is a 1-connected differential graded algebra. Define a multiplication in D in following way:

$$(a,c)\cdot(a',c') = (aa',cc') \in D^{n+m}$$

where $(a,c) \in D^n$ and $(a',c') \in D^m$. Let

$$d_A^n : A^n \to A^{n+1} \quad \text{and} \quad d_C^n : C^n \to C^{n+1},$$

$$d_A = \underset{n\geq 0}{\oplus} d_A^n \quad \text{and} \quad d_C = \underset{n\geq 0}{\oplus} d_C^n.$$

Define $d_D^n : D^n \to D^{n+1}$ by the rule

$$d_D^n(a,c) = (d_A^n(a), d_C^n(c))$$

for $(a,c) \in D^n$. Let

$$d_D = \underset{n\geq 0}{\oplus} d_D^n.$$

Now for any $(a,c) \in D$,

$$\begin{aligned} d_D d_D(a,c) &= d_D(d_A(a), d_C(c)) \\ &= (d_A d_A(a), d_C d_C(c)) \\ &= (0,0). \end{aligned}$$

So d_D is a differential. Now for $(a_1,c_1) \in D^n$ and $(a_2,c_2) \in D^m$,

$$\begin{aligned} d_D((a_1,c_1)\cdot(a_2,c_2)) &= d_D(a_1a_2, c_1c_2) \\ &= (d_A(a_1a_2), d_C(c_1c_2)) \\ &= (d_A(a_1)\cdot a_2 + (-1)^n a_1 d_A(a_2), d_C(c_1)\cdot c_2 + (-1)^n c_1 d_C(c_2)) \\ &= (d_A(a_1), d_C(c_1))\cdot(a_2,c_2) + (-1)^n(a_1,c_1)(d_A(a_2), d_C(c_2)) \\ &= (d_D(a_1,c_1))\cdot(a_2,c_2) + (-1)^n(a_1,c_1)(d_D(a_2,c_2)). \end{aligned}$$

So d_D is a derivation. Thus D becomes a differential graded algebra.

Next we show that D is simply connected, that is, $H^0(D) = \mathbb{Q}$ and $H^1(D) = 0$. Now

$$\begin{aligned} H^0(D) &= Z^0(D)/B^0(D) \\ &= Z^0(D) \\ &= \{(a,c) \in Z^0(A) \times Z^0(C) : f^0(a) = s^0(c)\}. \end{aligned}$$

Also $1_A \in A^0$ and $1_c \in C^0$. Then

$$d_D^0(1_A, 1_C) = (d_A 1_A, d_C 1_C) = (0,0)$$

implies $(1_A, 1_c) \in Z^0(D)$. Since A and C are 1-connected, we have

$$H^0(A) = \mathbb{Q} \quad \text{and} \quad H^0(C) = \mathbb{Q},$$

that is,

$$Z^0(A) = \mathbb{Q} \quad \text{and} \quad Z^0(C) = \mathbb{Q}.$$

Thus

$$(a, c) \in H^0(D) = Z^0(D) \subset Z^0(A) \times Z^0(C)$$

if and only if

$$a = r1_A \quad \text{and} \quad c = r1_C$$

for some $r \in \mathbb{Q}$. So $H^0(D) = \mathbb{Q}$. Again let $[(a, c)] \in H^1(D)$. Then $(a, c) \in Z^1(D)$. This implies

$$a \in Z^1(A), \quad c \in Z^1(C) \quad \text{and} \quad f^1(a) = s^1(c).$$

Since $a \in Z^1(A)$ and $c \in Z^1(C)$, we have $d_A^1(a) = 0$ and $d_C^1(a) = 0$ respectively. As A and C are 1-connected, we have

$$H^1(A) = 0 \quad \text{and} \quad H^1(C) = 0,$$

that is,

$$Z^1(A)/B^1(A) = B^1(A) \quad \text{and} \quad Z^1(C)/B^1(C) = B^1(C)$$

which implies $a = d_A^0(a')$ where $a' \in A^0$ and $c = d_C^0(c')$ where $c' \in C^0$. Now

$$f^1(a) = s^1(c),$$

that is,

$$f^1(d_A^0(a')) = s^1(d_C^0(c')).$$

Thus

$$d_B^0 f^0(a') = d_B^0 s^0(c'),$$

that is,

$$f^0(a') - s^0(c') \in ker d_B^0.$$

But $s^0 \in S$ deduces $(s^0)^* : H^0(C) \to H^0(B)$ is an isomorphism, that is, $(s^0)^*$: ker $d_C^0 \to$ ker d_B^0 is an isomorphism. Then there exists an element $c'' \in$ ker d_C^0 such that

$$s^0(c'') = f^0(a') - s^0(c'),$$

that is,

$$f^0(a') = s^0(c' + c'').$$

Thus $(x', c' + c'') \in D^0$ and

$$\begin{aligned} d_D^0(x', c' + c'') &= (d_A^0(a'), d_C^0(c') + d_C^0(c'')) \\ &= (d_A^0(a'), d_C^0(c')) \\ &= (a, c). \end{aligned}$$

This implies $(a, c) \in B^1(D)$. So $H^1(D) = 0$. Thus D is 1-connected.

The only thing left to show is $t \in S$, i.e., we have to show

$$t^* : H^*(D) \to H^*(A)$$

is an isomorphism. Let $F = \text{Ker}\ g$. Then from the following commutative diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ D \arrow[r, "t"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

in $\mathscr{D}$ we will have the following commutative diagram [40]

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}(C) \arrow[r] \arrow[d, "s^*"'] & H^n(F) \arrow[r] \arrow[d, equal] & H^n(D) \arrow[r] \arrow[d, "t^*"] & {} \\ \cdots \arrow[r] & H^{n-1}(B) \arrow[r] & H^n(F) \arrow[r] & H^n(A) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} H^n(C) \arrow[r] \arrow[d, "s^*"'] & H^{n+1}(F) \arrow[r] \arrow[d, equal] & \cdots \\ H^n(B) \arrow[r] & H^{n+1}(F) \arrow[r] & \cdots \end{tikzcd}$$

By Five Lemma, t^* is an isomorphism, showing $t \in S$.

Next for any differential graded algebra

$$E = \bigoplus_{n \geq 0} E^n$$

and differential graded algebra homomorphisms

$$u = \bigoplus_{n \geq 0} u^n : E \to A$$

and

$$v = \bigoplus_{n \geq 0} v^n : E \to C$$

in $\mathscr{D}$ the following diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "v"'] & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

commutes, that is, $fu = sv$. Consider the diagram:

$$\begin{tikzcd} E \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & D \arrow[r, "t"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ & C \arrow[r, "s"'] & B \end{tikzcd}$$

Define the map

$$\theta = \bigoplus_{n \geq 0} \theta^n : E \to D$$

by the rule

$$\theta(x) = (u(x), v(x))$$

for $x \in E$. Clearly, θ is well defined and also a d.g.a. homomorphism. Next for any $x \in E$,

$$t\theta(x) = t(u(x), v(x)) = u(x)$$

and

$$g\theta(x) = g(u(x), v(x)) = v(x).$$

So $t\theta = u$ and $g\theta = v$, that is, the two triangles are commutative. □

For the category $\mathscr{D}$ along with this set of morphisms S, the going with reliably holds.

Proposition 5.2.3. *If each $s_i : X_i \to Y_i$, $i \in I$ is an element of S where the index set I is an element of $\mathscr{U}$, then*

$$\bigwedge_{i \in I} s_i : \bigwedge_{i \in I} X_i \to \bigwedge_{i \in I} Y_i$$

is an element S.

Proof. The proof is obvious. □

The following result follows trivially.

Proposition 5.2.4. *The category $\mathscr{D}$ is complete.*

By considering all the above results, from Theorem 1.4.3 and Theorem 1.5.4 we can reach at the outcome that the Adams cocompletion of an object always exists.

Theorem 5.2.5. *Every object A of the category $\mathscr{D}$ has an Adams cocompletion A_S with respect to the set of morphisms S and there exists a morphism $e : A_S \to A$ in S which is couniversal with respect to the morphisms in S, that is, given a morphism $s : B \to A$ in S there exists a unique morphism $t : A_S \to B$ in S such that $st = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} A_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & A \\ B \arrow[ur, "s"'] & \end{tikzcd}$$

5.3 The result

In this section, minimal model for a simply connected differential graded algebra is obtained as the Adams cocompletion of that simply connected differential graded algebra.

Theorem 5.3.1. *Let A be a simply connected differential graded algebra and $M = M_A$ be the minimal model for A. Then $M \simeq A_S$.*

Proof. Let $e : A_S \to A$ be the morphism as defined in Theorem 5.2.5. we conclude from the couniversal property of e that there exists a unique d.g.a. map $\psi : A_S \to M$ such that $\rho\psi = e$.

$$\begin{tikzcd} A_S \arrow[r, "e"] \arrow[d, dashed, "\psi"'] & A \\ M \arrow[ur, "\rho"'] & \end{tikzcd}$$

Next we have the d.g.a. map $\rho : M \to A$ which induces an isomorphism on cohomology in all dimension. By Theorem 5.1.14, there exists a d.g.a. map $\varphi : M \to A_S$ such that $e\varphi \simeq \rho$.

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, dashed, "\varphi"'] & A \\ A_S \arrow[ur, "e"'] & \end{tikzcd}$$

From the following diagram

$$\begin{tikzcd}
A_S \arrow[r, "e"] \arrow[d, "\psi"'] \arrow[dd, bend right, "1_{A_S}"'] & A \\
M \arrow[d, "\varphi"'] & \\
A_S \arrow[uur, "e"'] &
\end{tikzcd}$$

we have

$$e\varphi\psi \simeq \rho\psi = e.$$

Then

$$(e\varphi\psi)^* = e^*,$$

that is,

$$e^*\varphi^*\psi^* = e^*$$

which implies $\varphi^*\psi^* = 1_{A_S}^*$. So $\varphi\psi \simeq 1_{A_S}$.

Again from the following diagram

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\varphi"'] \arrow[dd, bend right, "1_M"'] & A \\
A_S \arrow[d, "\psi"'] & \\
M \arrow[uur, "\rho"'] &
\end{tikzcd}$$

we have

$$\rho\psi\varphi = e\varphi \simeq \rho.$$

Then

$$(\rho\psi\varphi)^* = \rho^*,$$

that is,

$$\rho^*\psi^*\varphi^* = \rho^*$$

which implies $\psi^*\varphi^* = 1_M^*$. So $\psi\varphi \simeq 1_M$.

Now we have $\psi\varphi \simeq 1_M$ and $\varphi\psi \simeq 1_{A_S}$. Hence $M \simeq A_S$. $\square$

Chapter 6

Adams Cocompletion of a Graph

Numerous structures in real world can be represented on paper by means of a diagram comprising of a set of points together with lines joining some or all pairs of these points. A mathematical abstraction of such structures including points and lines drives us to the idea of graphs. The origins of graph theory can be taken from Euler's work on the Konigsberg bridge problem. The theory of graphs has established itself as a standout amongst the most rapidly growing areas of mathematics with many applications in various fields such as computer science, chemistry, engineering, social sciences etc,. Lately the category-theoretical approach to graph theory has become one of the most interesting area of study.

Given any graph G there exists a connected graph H, the center of which is isomorphic to G, is a renowned fact in the field of graph theory [20, 41]. In this chapter we will establish that the center $Z(H)$ of H is Adams cocompletion of the given graph G.

6.1 Result related to a graph

In this section we review some basics of graph theory and a result of Kopylov and Timofeev.

Definition 6.1.1. [42, 43] A *graph* $G = (V, E)$ consists of two sets V and E where

- the elements of V are called vertices (or nodes),
- the elements of E are called edges,
- each edge has a set of one or two vertices associated to it, which are called its endpoints. An edge is said to join its endpoints.

Throughout this chapter, denote by $V(G)$ and $E(G)$ to be the vertex set and edge set of a graph G respectively.

Definition 6.1.2. [44] A *graph homomorphism* f from a graph $G = (V(G), E(G))$ to a graph $G' = (V(G'), E(G'))$, denoted as

$$f : G \to G',$$

is a map

$$f : V(G) \to V(G')$$

that keeps adjacency, that is, $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(G')$.

The interpretation of isomorphism basically means that the two graphs are similar.

Definition 6.1.3. [44] Two graphs

$$G = (V(G), E(G))$$

and

$$G' = (V(G'), E(G'))$$

are called *isomorphic* if there exists a one-to-one correspondence between vertex sets $V(G)$ and $V(G')$ such that any two vertices are adjacent in G if and only if their images in the correspondence are adjacent in G'.

Definition 6.1.4. [44] A graph in which all vertices can be numbered (ordered from left to right) $x_1, x_2, \cdots, x_n$ in such a way that there is precisely one edge connecting every two consecutive vertices and there are no other edges, is called a *path*. Generally, any path connecting vertices x and y is called (x, y)-path.

Definition 6.1.5. [44] A graph is called *connected* if any two vertices in it are connected by some path; otherwise it is called *disconnected*.

Definition 6.1.6. [44] Let $G = (V(G), E(G))$ be a graph and $u, v \in V(G)$. The *distance* from u to v denoted by $d(u, v)$ is the length of the shortest (u, v)-path. Let $N_\infty(u)$ denote the set of farthest vertices from vertex u, that is, if $v \in N_\infty(u)$ and $w \notin N_\infty(u)$, then $d(u, w) < d(u, v)$. The distance between vertex u and set $N_\infty(u)$ is called the *eccentricity* of u.

Definition 6.1.7. [44] The *center* of G is a set of vertices of minimum eccentricity.

The following is an eminent result given by Kopylov and Timofeev.

Theorem 6.1.8. [20] *For any graph G, there exists a connected graph H such that the center of H is isomorphic to G.*

Let us denote the center of the graph H as $Z(H)$ and α be the isomorphism from $Z(H)$ to G. Applying the above result of Kopylov and Timofeev we prove the following result.

Theorem 6.1.9. *If K is a graph and $\beta : K \to G$ is an isomorphism, then there exists a unique isomorphism $\gamma : Z(H) \to K$ such that $\beta\gamma = \alpha$, that is, the following diagram commutes:*

$$\begin{tikzcd} Z(H) \arrow[r, "\alpha"] \arrow[d, dashed, "\gamma"'] & G \\ K \arrow[ur, "\beta"'] & \end{tikzcd}$$

Proof. Let us define the map $\gamma : V(Z(H)) \to V(K)$ (means $\gamma : Z(H) \to K$) by the rule

$$\gamma(x) = \beta^{-1}\alpha(x)$$

for every $x \in V(Z(H))$. Clearly, the map is well defined and is a graph isomorphism. Next for any $x \in V(Z(H))$,

$$\beta\gamma(x) = \beta\beta^{-1}\alpha(x) = \alpha(x)$$

shows that the diagram is commutative. The only thing left to show is the uniqueness of γ. Suppose that there exists another $\gamma' : V(Z(H)) \to V(K)$ satisfying $\beta\gamma' = \alpha$. Then for every $x \in V(Z(H))$,

$$\gamma(x) = \beta^{-1}\alpha(x) = \beta^{-1}\beta\gamma'(x) = \gamma'(x),$$

showing γ is unique. □

6.2 The category of graphs and graph homomorphisms

Let $\mathscr{G}$ denote the category of graphs and graph homomorphisms where every element of $\mathscr{G}$ is an element of $\mathscr{U}$. We fix a suitable set of graph homomorphisms in $\mathscr{G}$ as follows. Let S be a set of all graph homomorphisms $f : A \to B$ in $\mathscr{G}$ such that f is an isomorphism. For this chosen set of graph homomorphisms, the following result evidently follows from Theorem 1.5.2.

Proposition 6.2.1. *S is saturated.*

The following result holds for the set of graph homomorphisms S for the category $\mathscr{G}$.

Proposition 6.2.2. *If each $s_i : A_i \to B_i$ for $i \in I$ is an element of S where the index set I is an element of $\mathscr{U}$, then*

$$\bigwedge_{i\in I} s_i : \bigwedge_{i\in I} A_i \to \bigwedge_{i\in I} B_i$$

is an element in S.

Proof. Tensor product of graphs is the product in the category of graphs and graph homomorphisms. Let

$$V(A) = V(\prod_{i\in I} A_i) = \prod_{i\in I} V(A_i)$$

and

$$V(B) = V(\prod_{i\in I} B_i) = \prod_{i\in I} V(B_i).$$

Define a map

$$s = \underset{i\in I}{\wedge} s_i : V(A) \to V(B)$$

by the rule

$$s(a) = (s_i(a_i))_{i\in I}$$

for $a = (a_i)_{i\in I}$. Clearly, s is well defined. Let us consider $\{(a_i)_{i\in I}, (a'_i)_{i\in I}\} \in E(A)$. That means $(a_i)_{i\in I}$ and $(a'_i)_{i\in I}$ are adjacent in A, that is, a_i is adjacent with a'_i in A_i for each $i \in I$. Thus $s_i(a_i)$ is adjacent with $s_i(a'_i)$ in B_i for each $i \in I$, that is, $(s_i(a_i))_{i\in I}$ is adjacent with $(s_i(a'_i))_{i\in I}$ in B. Therefore, $\{s((a_i)_{i\in I}), s((a'_i)_{i\in I})\} \in E(B)$; this implies s is a graph homomorphism.

Let $(a_i)_{i\in I}$ and $(a'_i)_{i\in I} \in V(A)$ with $s((a_i)_{i\in I}) = s((a'_i)_{i\in I})$. So $s_i(a_i) = s_i(a'_i)$ for each $i \in I$ implies $a_i = a'_i$ for each $i \in I$; this results in $(a_i)_{i\in I} = (a'_i)_{i\in I}$ and hence s is injective. Next consider an element $(b_i)_{i\in I} \in V(B)$. As s_i is surjective, there exists some $a_i \in A_i$ such that $s_i(a_i) = b_i$ for each $i \in I$. Thus

$$(b_i)_{i\in I} = (s_i(a_i))_{i\in I} = s((a_i)_{i\in I})$$

showing s is surjective. Let $(a_i)_{i\in I}$ and $(a'_i)_{i\in I}$ be adjacent in A. Then $s((a_i)_{i\in I})$ and $s((a'_i)_{i\in I})$ are adjacent in B (already shown). Conversely, let $(a_i)_{i\in I}$ and $(a'_i)_{i\in I} \in V(A)$ and $s((a_i)_{i\in I})$ and $s((a'_i)_{i\in I})$ be adjacent in B. That means $(s_i(a_i))_{i\in I}$ and $(s_i(a'_i))_{i\in I}$ are adjacent in B; i.e., $s_i(a_i)$ is adjacent with $s_i(a'_i)$ in B_i for each $i \in I$ and hence a_i is adjacent with a'_i in A_i for each $i \in I$ (since s_i is an isomorphism). So $(a_i)_{i\in I}$ and $(a'_i)_{i\in I}$ are adjacent in A. Thus s is an isomorphism, that is, $s \in S$. □

We will exhibit that the set of graph homomorphisms S for the category $\mathscr{G}$ admits a calculus of right fractions.

Proposition 6.2.3. *S admits a calculus of right fractions.*

Proof. Since S is the set of all isomorphisms, it is clearly a closed family of morphisms of the category $\mathscr{G}$. Next we shall verify conditions (i) and (ii) of Theorem 1.2.4. In order to show condition (i) of Theorem 1.2.4 we have to show for morphisms $u : A \to B$ and $v : B \to C$ of $\mathscr{D}$, $vu \in S$ and $v \in S$ implies $u \in S$. As vu is a monomorphism, u is a monomorphism. Next for any $b \in B$, $v(b) \in C$. Since vu is an epimorphism, hence there exists some $a \in A$ such that $vu(a) = v(b)$ implying $u(a) = b$ (v is a monomorphism) which means u is an epimorphism. Thus $u \in S$.

For showing condition (ii) of Theorem 1.2.4 consider the following diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

in $\mathscr{G}$ with $s \in S$. We claim that the above diagram can be embedded to a weak pull-back diagram

$$\begin{tikzcd} D \arrow[r, dashed, "t"] \arrow[d, dashed, "g"'] & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

in $\mathscr{G}$ with $t \in S$. This is indeed represented by the following diagram:

$$\begin{tikzcd} V(D) \arrow[r, dashed, "t"] \arrow[d, dashed, "g"'] & V(A) \arrow[d, "f"] \\ V(C) \arrow[r, "s"'] & V(B) \end{tikzcd}$$

We construct a set $V(D)$ as follows:

$$\begin{aligned} V(D) &= \{(a, c) \in V(A) \times V(C) : f(a) = s(c)\} \\ &\subset V(A) \times V(C). \end{aligned}$$

Define $t : V(D) \to V(A)$ by the rule

$$t(a, c) = a$$

for $a \in V(A)$ and $g : V(D) \to V(C)$ by the rule

$$g(a, c) = c$$

for $c \in V(C)$. Clearly, the two maps are well defined. Let $\{(a, c), (a', c')\} \in E(D)$. That means (a, c) and (a', c') are adjacent in D. Then from the definition of $V(D)$ we can deduce that a is adjacent with a' and c is adjacent with c'. So $\{a, a'\} \in E(A)$. Then

$$\{t(a, c), t(a', c')\} = \{a, a'\} \in E(A).$$

So t is a graph homomorphism. Similarly, g is also a graph homomorphism. For all $(a, c) \in V(D)$,

$$ft(a, c) = f(a) = s(c) = sg(a, c),$$

that is, the above diagram commutes.

Let $u : V(E) \to V(A)$ and $v : V(E) \to V(C)$ in $\mathscr{G}$ be two morphisms such that $fu = sv$.

$$\begin{tikzcd}
V(E) \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& V(D) \arrow[r, "t"] \arrow[d, "g"'] & V(A) \arrow[d, "f"] \\
& V(C) \arrow[r, "s"'] & V(B)
\end{tikzcd}$$

Define $\theta : E \to D$ by the rule

$$\theta(x) = (u(x), v(x))$$

for $x \in E$. Clearly, θ is well defined and also a graph homomorphism. Next we show that the two triangles are commutative. Now for any $x \in X$

$$t\theta(x) = t(u(x), v(x)) = u(x)$$

and

$$g\theta(x) = g(u(x), v(x)) = v(x).$$

So $t\theta = u$ and $g\theta = v$.

Next consider (a, c) and $(a', c') \in V(D)$ with $t(a, c) = t(a', c')$; this implies $a = a'$. Since (a, c) and $(a', c') \in V(D)$, we have

$$f(a) = s(c) \text{ and } f(a') = s(c').$$

Then $s(c) = s(c')$ implies $c = c'$. So $(a, c) = (a', c')$, showing t is injective. Consider an element $a \in V(A)$; so $f(a) \in V(B)$. There exists some $c \in V(C)$ such that $s(c) = f(a)$. Clearly, $(a, c) \in V(D)$ such that $t(a, c) = a$ showing that t is surjective. Thus t is bijective. Let (a, c) and $(a', c') \in V(D)$ be adjacent in D. So $t(a, c) = a$ and $t(a', c') = a'$ are adjacent in A (since t is a homomorphism). Conversely, let (a, c) and $(a', c') \in V(D)$ and $t(a, c) = a$, $t(a', c') = a'$ be adjacent in A. So $f : A \to B$, being a homomorphism, deduces that $f(a)$ and $f(a')$ are adjacent in B, that is, $s(c)$ and $s(c')$ are adjacent in B. Thus c and c' are adjacent in C. So (a, c) and (a', c') are adjacent in D (by definition of $V(D)$). Therefore, t is an isomorphism, showing $t \in S$. □

The proof of the following result is trivial.

Proposition 6.2.4. *The category $\mathscr{G}$ is complete.*

By considering all the above results, from Theorem 1.4.3 and Theorem 1.5.4 we can reach at the outcome that the Adams cocompletion of an object always exists.

Theorem 6.2.5. *Every object G of the category $\mathscr{G}$ has an Adams cocompletion G_S with respect to the set of morphisms S. Furthermore, there exists a morphism $e : G_S \to G$ in S which is couniversal with respect to morphisms of S : given a morphism $s : K \to G$ in S there exists a unique morphism $t : G_S \to K$ in S such that $st = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} G_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & G \\ K \arrow[ur, "s"'] & \end{tikzcd}$$

6.3 $Z(H)$ as Adams cocompletion of G

In this section, we will show that $Z(H)$, the center of the graph H (as defined above), is the Adams cocompletion of the graph G.

Theorem 6.3.1. $Z(H) \cong G_S$.

Proof. Let us consider the following diagram:

$$\begin{tikzcd} Z(H) \arrow[r, "\alpha"] \arrow[d, dashed, "\gamma"'] & G \\ G_S \arrow[ur, "e"'] & \end{tikzcd}$$

By Theorem 6.1.9, we conclude that there exists a unique graph homomorphism $\gamma : Z(H) \to G_S$ in S such that $e\gamma = \alpha$.

Consider another diagram as follows:

$$\begin{tikzcd} G_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & G \\ Z(H) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

By Theorem 6.2.5, there exists a unique graph homomorphism $t : G_S \to Z(H)$ in S such that $\alpha t = e$.

From the following diagram

$$
\begin{tikzcd}
Z(H) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dd, bend right=60, "1_{Z(H)}"'] & G \\
G_S \arrow[d, "t"'] & \\
Z(H) \arrow[uur, "\alpha"'] &
\end{tikzcd}
$$

we conclude that

$$\alpha t\gamma = e\gamma = \alpha$$

and the uniqueness condition of α exhibits

$$t\gamma = 1_{Z(H)}.$$

From the following

$$
\begin{tikzcd}
G_S \arrow[r, "e"] \arrow[d, "t"'] \arrow[dd, bend right=60, "1_{G_S}"'] & G \\
Z(H) \arrow[d, "\gamma"'] & \\
G_S \arrow[uur, "e"'] &
\end{tikzcd}
$$

we have

$$e\gamma t = \alpha t = e$$

and uniqueness condition of e concludes that

$$\gamma t = 1_{G_S}.$$

Now we have

$$t\gamma = 1_{Z(H)} \text{ and } \gamma t = 1_{G_S};$$

this shows that $Z(H) \cong G_S$, that is, $Z(H)$ is the Adams cocompletion of G with respect to S. $\square$

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